Problem 1

Let \( n \) be a positive integer. Let \( T \) be the set of points \((x, y)\) in the plane where \( x \) and \( y \) are non-negative integers and \( x + y < n \). Each point of \( T \) is coloured red or blue. If a point \((x, y)\) is red, then so are all points \((x', y')\) of \( T \) with both \( x' \leq x \) and \( y' \leq y \). Define an \( X \)-set to be a set of \( n \) blue points having distinct \( x \)-coordinates, and a \( Y \)-set to be a set of \( n \) blue points having distinct \( y \)-coordinates. Prove that the number of \( X \)-sets is equal to the number of \( Y \)-sets.

Problem 2

Let \( BC \) be a diameter of the circle \( \Gamma \) with centre \( O \). Let \( A \) be a point on \( \Gamma \) such that \( 0^\circ < \angle AOB < 120^\circ \). Let \( D \) be the midpoint of the arc \( AB \) not containing \( C \). The line through \( O \) parallel to \( DA \) meets the line \( AC \) at \( J \). The perpendicular bisector of \( OA \) meets \( \Gamma \) at \( E \) and at \( F \). Prove that \( J \) is the incentre of the triangle \( CEF \).

Problem 3

Find all pairs of integers \( m, n \geq 3 \) such that there exist infinitely many positive integers \( a \) for which

\[
\frac{a^n + a - 1}{a^n + a^2 - 1}
\]

is an integer.

Problem 4

Let \( n \) be an integer greater than 1. The positive divisors of \( n \) are \( d_1, d_2, \ldots, d_k \) where \( 1 = d_1 < d_2 < \cdots < d_k = n \). Define \( D = d_1d_2 + d_2d_3 + \cdots + d_{k-1}d_k \).

(a) Prove that \( D < n^2 \).

(b) Determine all \( n \) for which \( D \) is a divisor of \( n^2 \).

Problem 5

Find all functions \( f \) from the set \( r \) of real numbers to itself such that

\[
(f(x) + f(z)) (f(y) + f(t)) = f(xy - zt) + f(xt + yz)
\]

for all \( x, y, z, t \) in \( r \).

Problem 6

Let \( \Gamma_1, \Gamma_2, \ldots, \Gamma_n \) be circles of radius 1 in the plane, where \( n \geq 3 \). Denote their centres by \( O_1, O_2, \ldots, O_n \) respectively. Suppose that no line meets more than two of the circles. Prove that

\[
\sum_{1 \leq i < j \leq n} \frac{1}{O_i O_j} \leq \frac{(n-1)\pi}{4}.
\]
