Contents

Chapter 1

Problems

1.1 Algebra

A1. Let T denote the set of all ordered triples (p,q,r) of nonnegative integers. Find all functions $f : T \to \mathbb{R}$ such that

$$f(p,q,r) = \begin{cases} 0 & \text{if } pqr = 0, \\ 1 + \frac{1}{6}\{f(p+1,q-1,r) + f(p-1,q+1,r) & \\ +f(p-1,q,r+1) + f(p+1,q,r-1) & \\ +f(p,q+1,r-1) + f(p,q-1,r+1)\} & \text{otherwise.} \end{cases}$$

A2. Let $a_0, a_1, a_2, \ldots$ be an arbitrary infinite sequence of positive numbers. Show that the inequality $1 + a_n > a_{n-1}\sqrt[n]{2}$ holds for infinitely many positive integers n.

A3. Let $x_1, x_2, \ldots, x_n$ be arbitrary real numbers. Prove the inequality

$$\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^2+x_2^2} + \cdots + \frac{x_n}{1+x_1^2+\cdots+x_n^2} < \sqrt{n}.$$

A4. Find all functions $f : \mathbb{R} \to \mathbb{R}$, satisfying

$$f(xy)(f(x) - f(y)) = (x - y)f(x)f(y)$$

for all x, y.

A5. Find all positive integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{99}{100} = \frac{a_0}{a_1} + \frac{a_1}{a_2} + \cdots + \frac{a_{n-1}}{a_n},$$

where $a_0 = 1$ and $(a_{k+1} - 1)a_{k-1} \geq a_k^2(a_k - 1)$ for $k = 1, 2, \ldots, n - 1$.

A6. Prove that for all positive real numbers a, b, c,

$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq 1.$$

1.2 Combinatorics

C1. Let $A = (a_1, a_2, \ldots, a_{2001})$ be a sequence of positive integers. Let m be the number of 3-element subsequences (a_i, a_j, a_k) with $1 \leq i < j < k \leq 2001$, such that $a_j = a_i + 1$ and $a_k = a_j + 1$. Considering all such sequences A, find the greatest value of m.

C2. Let n be an odd integer greater than 1 and let $c_1, c_2, \ldots, c_n$ be integers. For each permutation $a = (a_1, a_2, \ldots, a_n)$ of $\{1, 2, \ldots, n\}$, define $S(a) = \sum_{i=1}^{n} c_i a_i$. Prove that there exist permutations $a \neq b$ of $\{1, 2, \ldots, n\}$ such that $n!$ is a divisor of $S(a) - S(b)$.

C3. Define a *k-clique* to be a set of k people such that every pair of them are acquainted with each other. At a certain party, every pair of 3-cliques has at least one person in common, and there are no 5-cliques. Prove that there are two or fewer people at the party whose departure leaves no 3-clique remaining.

C4. A set of three nonnegative integers $\{x, y, z\}$ with $x < y < z$ is called *historic* if $\{z - y, y - x\} = \{1776, 2001\}$. Show that the set of all nonnegative integers can be written as the union of pairwise disjoint historic sets.

C5. Find all finite sequences $(x_0, x_1, \ldots, x_n)$ such that for every j, $0 \leq j \leq n$, x_j equals the number of times j appears in the sequence.

C6. For a positive integer n define a sequence of zeros and ones to be *balanced* if it contains n zeros and n ones. Two balanced sequences a and b are *neighbors* if you can move one of the $2n$ symbols of a to another position to form b. For instance, when $n = 4$, the balanced sequences 01101001 and 00110101 are neighbors because the third (or fourth) zero in the first sequence can be moved to the first or second position to form the second sequence. Prove that there is a set S of at most $\frac{1}{n+1}\binom{2n}{n}$ balanced sequences such that every balanced sequence is equal to or is a neighbor of at least one sequence in S.

C7. A pile of n pebbles is placed in a vertical column. This configuration is modified according to the following rules. A pebble can be moved if it is at the top of a column which contains at least two more pebbles than the column immediately to its right. (If there are no pebbles to the right, think of this as a column with 0 pebbles.) At each stage, choose a pebble from among those that can be moved (if there are any) and place it at the top of the column to its right. If no pebbles can be moved, the configuration is called a *final configuration*. For each n, show that, no matter what choices are made at each stage, the final configuration obtained is unique. Describe that configuration in terms of n.

Alternative Version. A pile of 2001 pebbles is placed in a vertical column. This configuration is modified according to the following rules. A pebble can be moved if it is at the top of a column which contains at least two more pebbles than the column immediately to its right. (If there are no pebbles to the right, think of this as a column with 0 pebbles.) At each stage, choose a pebble from among those that can be moved (if there are any) and place it at the top of the column to its right. If no pebbles can be moved, the configuration is called a *final configuration*. Show that, no matter what choices are made at each stage, the final configuration obtained is unique. Describe that configuration as follows: Determine the number, c, of nonempty columns, and for each $i = 1, 2, \ldots, c$, determine the number of pebbles p_i in column i, where column 1 is the leftmost column, column 2 the next to the right, and so on.

C8. Twenty-one girls and twenty-one boys took part in a mathematical competition. It turned out that

(a) each contestant solved at most six problems, and

(b) for each pair of a girl and a boy, there was at least one problem that was solved by both the girl and the boy.

Show that there is a problem that was solved by at least three girls and at least three boys.

1.3 Geometry

G1. Let A_1 be the center of the square inscribed in acute triangle ABC with two vertices of the square on side BC. Thus one of the two remaining vertices of the square is on side AB and the other is on AC. Points $B_1,\ C_1$ are defined in a similar way for inscribed squares with two vertices on sides AC and AB, respectively. Prove that lines $AA_1,\ BB_1,\ CC_1$ are concurrent.

G2. In acute triangle ABC with circumcenter O and altitude AP, $\angle C \geq \angle B + 30°$. Prove that $\angle A + \angle COP < 90°$.

G3. Let ABC be a triangle with centroid G. Determine, with proof, the position of the point P in the plane of ABC such that $AP{\cdot}AG+BP{\cdot}BG+CP{\cdot}CG$ is a minimum, and express this minimum value in terms of the side lengths of ABC.

G4. Let M be a point in the interior of triangle ABC. Let A' lie on BC with MA' perpendicular to BC. Define B' on CA and C' on AB similarly. Define

$$p(M) = \frac{MA' \cdot MB' \cdot MC'}{MA \cdot MB \cdot MC}.$$

Determine, with proof, the location of M such that $p(M)$ is maximal. Let $\mu(ABC)$ denote this maximum value. For which triangles ABC is the value of $\mu(ABC)$ maximal?

G5. Let ABC be an acute triangle. Let DAC, EAB, and FBC be isosceles triangles exterior to ABC, with $DA = DC, EA = EB$, and $FB = FC$, such that

$$\angle ADC = 2\angle BAC, \quad \angle BEA = 2\angle ABC, \quad \angle CFB = 2\angle ACB.$$

Let D' be the intersection of lines DB and EF, let E' be the intersection of EC and DF, and let F' be the intersection of FA and DE. Find, with proof, the value of the sum

$$\frac{DB}{DD'} + \frac{EC}{EE'} + \frac{FA}{FF'}.$$

G6. Let ABC be a triangle and P an exterior point in the plane of the triangle. Suppose AP, BP, CP meet the sides BC, CA, AB (or extensions thereof) in D, E, F, respectively. Suppose further that the areas of triangles PBD, PCE, PAF are all equal. Prove that each of these areas is equal to the area of triangle ABC itself.

G7. Let O be an interior point of acute triangle ABC. Let A_1 lie on BC with OA_1 perpendicular to BC. Define B_1 on CA and C_1 on AB similarly. Prove that O is the circumcenter of ABC if and only if the perimeter of $A_1B_1C_1$ is not less than any one of the perimeters of AB_1C_1, BC_1A_1, and CA_1B_1.

G8. Let ABC be a triangle with $\angle BAC = 60°$. Let AP bisect $\angle BAC$ and let BQ bisect $\angle ABC$, with P on BC and Q on AC. If $AB + BP = AQ + QB$, what are the angles of the triangle?

1.4 Number Theory

N1. Prove that there is no positive integer n such that, for $k = 1, 2, \ldots, 9$, the leftmost digit (in decimal notation) of $(n+k)!$ equals k.

N2. Consider the system

$$x + y = z + u$$
$$2xy = zu.$$

Find the greatest value of the real constant m such that $m \leq x/y$ for any positive integer solution (x, y, z, u) of the system, with $x \geq y$.

N3. Let $a_1 = 11^{11}$, $a_2 = 12^{12}$, $a_3 = 13^{13}$, and

$$a_n = |a_{n-1} - a_{n-2}| + |a_{n-2} - a_{n-3}|, \qquad n \geq 4.$$

Determine $a_{14^{14}}$.

N4. Let $p \geq 5$ be a prime number. Prove that there exists an integer a with $1 \leq a \leq p-2$ such that neither $a^{p-1} - 1$ nor $(a+1)^{p-1} - 1$ is divisible by p^2.

N5. Let $a > b > c > d$ be positive integers and suppose

$$ac + bd = (b + d + a - c)(b + d - a + c).$$

Prove that $ab + cd$ is not prime.

N6. Is it possible to find 100 positive integers not exceeding 25,000, such that all pairwise sums of them are different?

Chapter 2

Algebra

Problem A1. Let T denote the set of all ordered triples (p, q, r) of nonnegative integers. Find all functions $f : T \to \mathbb{R}$ such that

$$f(p,q,r) = \begin{cases} 0 & \text{if } pqr = 0, \\ 1 + \frac{1}{6}\{f(p+1,q-1,r) + f(p-1,q+1,r) & \\ +f(p-1,q,r+1) + f(p+1,q,r-1) & \\ +f(p,q+1,r-1) + f(p,q-1,r+1)\} & \text{otherwise.} \end{cases}$$

Solution. First, we will show that there is at most one function which satisfies the given conditions. Suppose that f_1 and f_2 are two such functions. Define $h = f_1 - f_2$. Then $h : T \to \mathbb{R}$ satisfies

$$h(p,q,r) = \begin{cases} 0 & \text{if } pqr = 0, \\ \frac{1}{6}\{h(p+1,q-1,r) + h(p-1,q+1,r) & \\ +h(p-1,q,r+1) + h(p+1,q,r-1) & \\ +h(p,q+1,r-1) + h(p,q-1,r+1)\} & \text{otherwise.} \end{cases}$$

Observe that the second condition states that $h(p, q, r)$ is equal to the average of the values of h at the six points $(p + 1, q - 1, r)$, etc., which are the vertices of a regular hexagon with center at (p, q, r) lying in the plane $x + y + z = p + q + r$. It suffices to show that $h = 0$ for all points in T. Let n be a positive integer. Consider the subset H

of the plane $x+y+z=n$ that lies in the "nonnegative" octant $\{(x,y,z) : x,y,z \geq 0\}$. Suppose h attains its maximum on $H \cap T$ at (p,q,r). If $pqr = 0$ then the maximum value for h on $H \cap T$ is 0. If $pqr \neq 0$, the averaging property of h implies that the values of h on the six points $(p+1, q-1, r)$, etc. are all equal to $h(p,q,r)$. (The six points are all in H). In particular, h also attains its maximum at $(p+1, q-1, r)$. Repeating the argument (if necessary) using $(p+1, q-1, r)$ as the center point,we see that

$$h(p,q,r) = h(p+1, q-1, r) = h(p+2, q-2, r).$$

Continuing this process, we conclude that $h(p,q,r) = h(p+q, 0, r) = 0$. Thus the maximum value of h on $H \cap T$ is 0. By applying the same argument to the function $-h = f_2 - f_1$, we see that the minimum value attained by h on $H \cap T$ is also 0. Thus $h = 0$ for all points in $H \cap T$. Varying n, we conclude that $h = 0$ on all points in T. We will complete the solution by noting that $f : T \to \mathbb{R}$ defined by

$$f(p,q,r) = \begin{cases} 0 & \text{if } pqr = 0, \\ \dfrac{3pqr}{p+q+r} & \text{otherwise} \end{cases}$$

satisfies both conditions of the problem, and is the unique solution.

Remark 1. One can guess the solution function in the following way: For any function f defined on T, define the function $A[f]$ by

$$A[f](p,q,r) = \frac{1}{6}\left(f(p+1, q-1, r) + \cdots\right).$$

It is easy to check that if c is a constant, then

$$A[cf] = cA[f] \quad \text{and} \quad A[c+f] = c + A[f].$$

Also note that if h is defined by $h(p,q,r) = f(p,q,r)/(p+q+r)$, then

$$A[h](p,q,r) = \frac{A[f](p,q,r)}{p+q+r}.$$

We need to find a function f that satisfies the boundary conditions, as well as the second condition

$$f = A[f] + 1.$$

It is natural to start by considering $g(p, q, r) = pqr$, which satisfies the boundary conditions. We shall suitably modify this so that the second condition is also satisfied. Observe that

$$A[g](p, q, r) = \frac{1}{6}\left(6pqr - 2(p + q + r)\right) = g(p, q, r) - \frac{p + q + r}{3}.$$

Thus there is an extra term involving $p + q + r$. To take care of this, we divide pqr by $p + q + r$ and consider the function $u(p, q, r) = pqr/(p + q + r)$. We have

$$A[u](p, q, r) = \frac{A[g](p, q, r)}{p + q + r} = u(p, q, r) - \frac{1}{3}.$$

Thus

$$A[3u] = 3u - 1,$$

and hence $3pqr/(p + q + r)$ satisfies the second condition.

Remark 2. One can consider the two-dimensional version of this problem, where $f(p, q) = 0$ if $pq = 0$ and $f(p, q) = 1 + [f(p + 1, q - 1) + f(p - 1, q + 1)]/2$ otherwise. The unique solution is $f(p, q) = pq$.

Problem A2. Let $a_0, a_1, a_2, \ldots$ be an arbitrary infinite sequence of positive numbers. Show that the inequality $1 + a_n > a_{n-1}\sqrt[n]{2}$ holds for infinitely many positive integers n.

Solution 1. Let $c_0, c_1, c_2, c_3, \ldots$ be the sequence defined by $c_0 = 1$ and

$$c_n = \left(\frac{a_{n-1}}{1 + a_n}\right) c_{n-1}, \quad n \geq 1.$$

Rewriting this as $c_n = a_{n-1}c_{n-1} - a_n c_n$, we obtain the telescoping sum

$$c_1 + c_2 + \cdots + c_n = a_0 - a_n c_n. \tag{$*$}$$

The assertion of the problem is equivalent to: $c_n/c_{n-1} < 2^{-1/n}$ for infinitely many n. Assume to the contrary that there exists N such that the opposite inequality holds for all $n \geq N$. Then for $n > N$,

$$c_n \geq c_N \cdot 2^{-(\frac{1}{N+1} + \frac{1}{N+2} + \cdots + \frac{1}{n})} = C \cdot 2^{-(1 + \frac{1}{2} + \cdots + \frac{1}{n})},$$

where $C = c_N \cdot 2^{1 + \frac{1}{2} + \cdots + \frac{1}{N}}$ is a positive constant. If $2^{k-1} \leq n < 2^k$, then

$$\begin{aligned} 1 + \frac{1}{2} + \cdots + \frac{1}{n} &\leq 1 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \cdots + \frac{1}{7}\right) + \cdots + \left(\frac{1}{2^{k-1}} + \cdots + \frac{1}{2^k - 1}\right) \\ &\leq 1 + 1 + 1 + \cdots + 1 \\ &= k, \end{aligned}$$

so that

$$c_n \geq C \cdot 2^{-k} \quad \text{for} \quad 2^{k-1} \leq n < 2^k.$$

Let r be such that $2^{r-1} \leq N < 2^r$, and let $m > r$. Then

$$\begin{aligned} c_{2^r} + c_{2^r+1} + \cdots + c_{2^m-1} &= (c_{2^r} + \cdots + c_{2^{r+1}-1}) + (c_{2^{r+1}} + \cdots + c_{2^{r+2}-1}) \\ &\quad + \cdots + (c_{2^{m-1}} + \cdots + c_{2^m-1}) \\ &\geq C \cdot (2^r \cdot 2^{-r-1} + 2^{r+1} \cdot 2^{-r-2} + \cdots + 2^{m-1} \cdot 2^{-m}) \\ &= \frac{C \cdot (m - r)}{2}, \end{aligned}$$

showing that the sum of the c_n can be made arbitrarily large. However, by $(*)$, this sum can never exceed a_0. This contradiction shows that $c_n/c_{n-1} < 2^{-1/n}$ for infinitely many n, as desired.

Solution 2. Arguing by contradiction, suppose there is N such that $1+a_n \leq a_{n-1}2^{1/n}$ for $n \geq N$. Multiply both sides by

$$b_n = 2^{-(1+\frac{1}{2}+\cdots+\frac{1}{n})}$$

to get

$$b_n + A_n \leq A_{n-1},$$

where $A_n = b_n a_n$.

Thus we have

$$\begin{aligned} b_N &\leq A_{N-1} - A_N \\ b_{N+1} &\leq A_N - A_{N+1} \\ \vdots \quad &\vdots \qquad \vdots \\ b_n &\leq A_{n-1} - A_n, \end{aligned}$$

and thus

$$\sum_{j=N}^{n} b_j \leq A_{N-1} - A_n \leq A_{N-1},$$

since the a_j are positive.

We shall show, however, that

$$\sum_{n \geq N} b_n$$

diverges. To see this, note that because $1/x$ is monotone decreasing, a simple comparison of areas yields

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \int_1^n \frac{dx}{x} = \log n,$$

for any positive integer n. Hence

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}<1+\log n,$$

and

$$b_n > 2^{-1-\log n} = \frac{1}{2}\,n^{-\log 2} > \frac{1}{2n}.$$

Because the harmonic series diverges (which can be proven by comparing areas as above, or with more elementary and well-known arguments), $\sum_{n\geq N} b_n$ diverges as well.

Problem A3. Let $x_1, x_2, \ldots, x_n$ be arbitrary real numbers. Prove the inequality

$$\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^2+x_2^2} + \cdots + \frac{x_n}{1+x_1^2+\cdots+x_n^2} < \sqrt{n}.$$

Solution 1. By the Cauchy-Schwarz inequality,

$$a_1 + a_2 + \cdots + a_n \leq \sqrt{n}\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

for any real numbers $a_1, a_2, \ldots, a_n$. Taking $a_k = x_k/(1 + x_1^2 + \cdots + x_k^2)$ for $k = 1, 2, \cdots, n$, it suffices to prove that

$$\left(\frac{x_1}{1+x_1^2}\right)^2 + \left(\frac{x_2}{1+x_1^2+x_2^2}\right)^2 + \cdots + \left(\frac{x_n}{1+x_1^2+\cdots+x_n^2}\right)^2 < 1.$$

Observe that for $k \geq 2$,

$$\begin{aligned}\left(\frac{x_k}{1+x_1^2+\cdots+x_k^2}\right)^2 &= \frac{x_k^2}{(1+x_1^2+\cdots+x_k^2)^2} \\ &\leq \frac{x_k^2}{\left(1+x_1^2+\cdots+x_{k-1}^2\right)(1+x_1^2+\cdots+x_k^2)} \\ &= \frac{1}{\left(1+x_1^2+\cdots+x_{k-1}^2\right)} - \frac{1}{(1+x_1^2+\cdots+x_k^2)}.\end{aligned}$$

For $k = 1$, similar reasoning yields the inequality

$$\left(\frac{x_1}{1+x_1^2}\right)^2 \leq 1 - \frac{1}{1+x_1^2}.$$

Summing these inequalities, the right-hand side telescopes to yield

$$\sum_{k=1}^{n}\left(\frac{x_k}{1+x_1^2+\cdots+x_k^2}\right)^2 \leq 1 - \frac{1}{1+x_1^2+\cdots+x_n^2} < 1.$$

Solution 2. Let

$$a_n = \sup\left(\frac{x_1}{1+x_1^2} + \cdots + \frac{x_n}{1+x_1^2+\cdots+x_n^2}\right)$$

and

$$b_n(r) = \sup \left(\frac{x_1}{r^2 + x_1^2} + \cdots + \frac{x_n}{r^2 + x_1^2 + \cdots + x_n^2} \right),$$

where the supremums are taken over all real $x_1, \ldots, x_n$. Replacing x_i by rx_i in the second formula shows that $b_n(r) = a_n/r$ when $r > 0$. Hence splitting off all but the first term gives

$$a_n = \sup_{x_1} \left(\frac{x_1}{1 + x_1^2} + \frac{a_{n-1}}{\sqrt{1 + x_1^2}} \right).$$

The result now follows by induction once one shows $a_1 = 1/2 < 1$ and

$$\frac{x}{1 + x^2} + \frac{\sqrt{n}}{\sqrt{1 + x^2}} < \sqrt{n+1}.$$

This latter inequality can be proven as follows: Without loss of generality, let x be positive (the inequality obviously holds for $x = 0$ and negative x), and let $0 < \theta < \pi/2$ such that $\tan \theta = x$. Also choose $0 < \alpha < \pi/2$ such that $\tan \alpha = \sqrt{n}$. Then

$$\begin{aligned}
\frac{x}{1 + x^2} + \frac{\sqrt{n}}{\sqrt{1 + x^2}} &= \sin \theta \cos \theta + \sqrt{n} \cos \theta \\
&< \sin \theta + \sqrt{n} \cos \theta \\
&= \sqrt{n+1} \left(\frac{1}{\sqrt{n+1}} \sin \theta + \frac{\sqrt{n}}{\sqrt{n+1}} \cos \theta \right) \\
&= \sqrt{n+1} \, (\cos \alpha \sin \theta + \sin \alpha \cos \theta) \\
&= \sqrt{n+1} \sin(\theta + \alpha) \\
&\leq \sqrt{n+1}.
\end{aligned}$$

Problem A4. Find all functions $f : \mathbb{R} \to \mathbb{R}$, satisfying

$$f(xy)(f(x) - f(y)) = (x - y)f(x)f(y)$$

for all x, y.

Solution. We wish to find all real-valued functions with real domain satisfying

$$f(xy)(f(x) - f(y)) = (x - y)f(x)f(y) \tag{1}$$

for all real x, y. Substituting $y = 1$ into (1) yields

$$f(x)^2 = xf(x)f(1). \tag{2}$$

If $f(1) = 0$, then $f(x) = 0$ for all x. This satisfies (1), yielding one solution. Suppose then, that $f(1) = C \neq 0$. Equation (2) implies that $f(0) = 0$. Now let G be a set of points x for which $f(x) \neq 0$. By (2),

$$f(x) = xf(1) \quad \text{for all} \quad x \in G.$$

Hence (1) can be satisfied only by functions satisfying

$$f(x) = \begin{cases} Cx & \text{if } x \in G, \\ 0 & \text{if } x \notin G. \end{cases} \tag{3}$$

We must determine the structure of G so that the function defined by (3) satisfies (1) for all real x, y. It is easy to check that if $x \neq y$ and both x and y are elements of G, then the function defined by (3) satisfies (1) if and only if $xy \in G$. If neither x nor y are elements of G then (1) is satisfied. By symmetry, the only other case to look at is $x \in G, y \notin G$. In this case, (1) implies that

$$f(xy)f(x) = 0,$$

which in turn implies that $f(xy) = 0$. Thus:

$$\text{If} \quad x \in G, y \notin G, \quad \text{then} \quad xy \notin G. \tag{4}$$

This implies the following facts about G:

(a) If $x \in G$, then $1/x \in G$. This is true, for otherwise (4) forces $1 \notin G$, which is impossible (recall that we are assuming that $f(1) \neq 0$, so $1 \in G$).

(b) If $x, y \in G$, then $xy \in G$. By (a) above, $1/x \in G$, so if $xy \notin G$, then (4) implies that $y = (xy)(1/x) \notin G$, a contradiction.

(c) If $x, y \in G$, then $x/y \in G$. This follows easily from (a) and (b).

Consequently, G is a set that contains 1, does not contain 0, and is closed under multiplication and division. It is easy to check that any such set will satisfy (a) above (since $1 \in G$) and (4): If G is closed under multiplication and division and $x \in G, y \notin G$, then $xy \notin G$, for otherwise, $y = (xy)/x \in G$, a contradiction.

Therefore, closure under multiplication and division completely characterizes G, and we can finally write the full answer to the problem:

$$f(x) = \begin{cases} Cx & \text{if } x \in G, \\ 0 & \text{if } x \notin G, \end{cases}$$

where C is an arbitrary fixed real number, and G is any subset of R that is closed under multiplication and division (i.e., any subgroup of the nonzero real numbers under multiplication). Note that $C = 0$ yields the "trivial" solution derived earlier.

Problem A5. Find all positive integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{99}{100} = \frac{a_0}{a_1} + \frac{a_1}{a_2} + \cdots + \frac{a_{n-1}}{a_n},$$

where $a_0 = 1$ and $(a_{k+1} - 1)a_{k-1} \geq a_k^2(a_k - 1)$ for $k = 1, 2, \ldots, n-1$.

Solution. Let $a_1, a_2, \ldots, a_n$ be positive integers satisfying the conditions of the problem. Then $a_k > a_{k-1}$, and hence $a_k \geq 2$ for $k = 1, 2, \ldots, n-1$. The inequality $(a_{k+1} - 1)a_{k-1} \geq a_k^2(a_k - 1)$ can be written in the form

$$\frac{a_{k-1}}{a_k} + \frac{a_k}{a_{k+1} - 1} \leq \frac{a_{k-1}}{a_k - 1}.$$

Summing these inequalities for $k = i+1, i+2, \ldots, n-1$, together with the obvious inequality $a_{n-1}/a_n < a_{n-1}/(a_n - 1)$, we obtain

$$\frac{a_i}{a_{i+1}} + \frac{a_{i+1}}{a_{i+2}} + \cdots + \frac{a_{n-1}}{a_n} < \frac{a_i}{a_{i+1} - 1}. \tag{$*$}$$

We now determine a_1, $a_2, \ldots, a_n$. Using the sum given in the problem statement and $(*)$, with $i = 0$, we obtain

$$\frac{1}{a_1} \leq \frac{99}{100} < \frac{1}{a_1 - 1},$$

so $a_1 = 2$. Using a similar approach with $i = 1$ we find

$$\frac{1}{a_2} \leq \frac{1}{a_1}\left(\frac{99}{100} - \frac{1}{a_1}\right) < \frac{1}{a_2 - 1},$$

and it follows that $a_2 = 5$. Repeating this argument with $i = 2$ and then $i = 3$ we obtain

$$\frac{1}{a_3} \leq \frac{1}{a_2}\left(\frac{99}{100} - \frac{1}{a_1} - \frac{a_1}{a_2}\right) < \frac{1}{a_3 - 1},$$

from which $a_3 = 56$, and

$$\frac{1}{a_4} \leq \frac{1}{a_3}\left(\frac{99}{100} - \frac{1}{a_1} - \frac{a_1}{a_2} - \frac{a_2}{a_3}\right) < \frac{1}{a_4 - 1},$$

which implies that $a_4 = 25{\cdot}56^2 = 78400$. Continuing with the argument to determine a_5 we find

$$\frac{1}{a_5} \leq \frac{1}{a_4}\left(\frac{99}{100} - \frac{1}{2} - \frac{2}{5} - \frac{5}{56} - \frac{56}{25{\cdot}56^2}\right) = 0,$$

which is impossible. It is easy to verify that the positive integers $a_1 = 2$, $a_2 = 5$, $a_3 = 56$, $a_4 = 25{\cdot}56^2$ satisfy the conditions of the problem. The preceding argument shows that the solution is unique.

Problem A6. Prove that for all positive real numbers a, b, c,

$$\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ab}} \geq 1.$$

Solution. First we shall prove that

$$\frac{a}{\sqrt{a^2+8bc}} \geq \frac{a^{\frac{4}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}},$$

or equivalently, that

$$\left(a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}\right)^2 \geq a^{\frac{2}{3}}(a^2+8bc).$$

The AM-GM inequality yields

$$\begin{aligned}\left(a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}\right)^2 - \left(a^{\frac{4}{3}}\right)^2 &= \left(b^{\frac{4}{3}}+c^{\frac{4}{3}}\right)\left(a^{\frac{4}{3}}+a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}\right)\\ &\geq 2b^{\frac{2}{3}}c^{\frac{2}{3}} \cdot 4a^{\frac{2}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}\\ &= 8a^{\frac{2}{3}}bc.\end{aligned}$$

Thus

$$\begin{aligned}\left(a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}\right)^2 &\geq \left(a^{\frac{4}{3}}\right)^2 + 8a^{\frac{2}{3}}bc\\ &= a^{\frac{2}{3}}(a^2+8bc),\end{aligned}$$

so

$$\frac{a}{\sqrt{a^2+8bc}} \geq \frac{a^{\frac{4}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}}.$$

Similarly, we have

$$\frac{b}{\sqrt{b^2+8ca}} \geq \frac{b^{\frac{4}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}} \quad \text{and}$$

$$\frac{c}{\sqrt{c^2+8ab}} \geq \frac{c^{\frac{4}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}}.$$

Adding these three inequalities yields

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\geq 1.$$

Comment. The proposer conjectures that for any $a, b, c > 0$ and $\lambda \geq 0$, the following inequality holds:

$$\frac{a}{\sqrt{a^2+\lambda bc}}+\frac{b}{\sqrt{b^2+\lambda ca}}+\frac{c}{\sqrt{c^2+\lambda ab}}\geq \frac{3}{\sqrt{1+\lambda}}.$$

Chapter 3

Combinatorics

Problem C1. Let $A = (a_1,\, a_2, \ldots, a_{2001})$ be a sequence of positive integers. Let m be the number of 3-element subsequences (a_i, a_j, a_k) with $1 \le i < j < k \le 2001$, such that $a_j = a_i + 1$ and $a_k = a_j + 1$. Considering all such sequences A, find the greatest value of m.

Solution. Consider the following two operations on the sequence A:

(1) If $a_i > a_{i+1}$, transpose these terms to obtain the new sequence $(a_1,\, a_2, \ldots, a_{i+1},\, a_i, \ldots, a_{2001})$.

(2) If $a_{i+1} = a_i + 1 + d$, where $d > 0$, increase $a_1, \ldots, a_i$ by d to obtain the new sequence $(a_1{+}d,\, a_2{+}d, \ldots, a_i{+}d,\, a_{i+1}, \ldots, a_{2001})$.

It is clear that performing operation (1) cannot reduce m. By applying (1) repeatedly, the sequence can be rearranged to be nondecreasing. Thus we may assume that our sequence for which m is maximal is nondecreasing. Next, note that if A is nondecreasing, then performing operation (2) cannot reduce the value of m. It follows that any A with maximum m is of the form

$$(\underbrace{a, \ldots, a}_{t_1},\, \underbrace{a{+}1, \ldots, a{+}1}_{t_2}, \ldots, \underbrace{a{+}s{-}1, \ldots, a{+}s{-}1}_{t_s})$$

where $t_1, \ldots, t_s$ are the number of terms in each subsequence, and $s \geq 3$. For such a sequence A,

$$m = t_1t_2t_3 + t_2t_3t_4 + \cdots + t_{s-2}t_{s-1}t_s. \qquad (*)$$

It remains to find the best choice of s and the best partition of 2001 into positive integers $t_1, \ldots, t_s$.

The maximum value of m occurs when $s = 3$ or $s = 4$. If $s > 4$ then we may increase the value given by $(*)$ by using a partition of 2001 into $s - 1$ parts, namely

$$t_2, t_3, (t_1 + t_4), \ldots, t_s.$$

Note that when $s = 4$ this modification does not change the value given by $(*)$. Hence the maximum value m can be obtained with $s = 3$. In this case, $m = t_1t_2t_3$ is largest when $t_1 = t_2 = t_3 = 2001/3 = 667$. Thus the maximum value of m is 667^3. This maximum value is attained when $s = 4$ as well, in this case for sequences with $t_1 = a, t_2 = t_3 = 667$, and $t_4 = 667 - a$, where $1 \leq a \leq 666$.

Problem C2. Let n be an odd integer greater than 1 and let $c_1, c_2, \ldots, c_n$ be integers. For each permutation $a = (a_1, a_2, \ldots, a_n)$ of $\{1, 2, \ldots, n\}$, define $S(a) = \sum_{i=1}^{n} c_i a_i$. Prove that there exist permutations $a \neq b$ of $\{1, 2, \ldots, n\}$ such that $n!$ is a divisor of $S(a) - S(b)$.

Solution. Let $\sum S(a)$ be the sum of $S(a)$ over all $n!$ permutations $a = (a_1, a_2, \ldots, a_n)$. We compute $\sum S(a) \mod n!$ two ways, one of which depends on the desired conclusion being false, and reach a contradiction when n is odd.

First way. In $\sum S(a)$, c_1 is multiplied by each $k \in \{1, \ldots, n\}$ a total of $(n-1)!$ times, once for each permutation of $\{1, \ldots, n\}$ in which $a_1 = k$. Thus the coefficient of c_1 in $\sum S(a)$ is

$$(n-1)!(1 + 2 + \cdots + n) = (n+1)!/2.$$

The same is true for all c_i, so

$$\sum S(a) = \frac{(n+1)!}{2} \sum_{i=1}^{n} c_i. \tag{1}$$

Second way. If $n!$ is not a divisor of $S(a) - S(b)$ for any $a \neq b$, then each $S(a)$ must have a different remainder mod $n!$. Since there are $n!$ permutations, these remainders must be precisely the numbers $0, 1, 2, \ldots, n!-1$. Thus

$$\sum S(a) \equiv \frac{(n! - 1)n!}{2} \mod n!. \tag{2}$$

Combining (1) and (2), we get

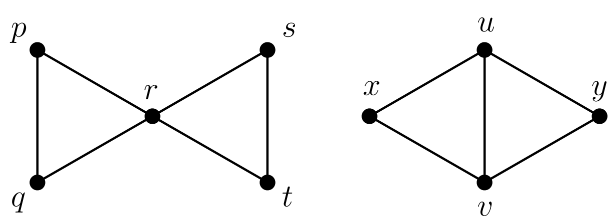
$$\frac{(n+1)!}{2} \sum_{i=1}^{n} c_i \equiv \frac{(n! - 1)n!}{2} \mod n!. \tag{3}$$

Now, for n odd, the left side of (3) is congruent to 0 modulo $n!$, while for $n > 1$ the right side is not congruent to 0 ($n! - 1$ is odd). For $n > 1$ and odd, we have a contradiction.

Problem C3. Define a *k-clique* to be a set of k people such that every pair of them are acquainted with each other. At a certain party, every pair of 3-cliques has at least one person in common, and there are no 5-cliques. Prove that there are two or fewer people at the party whose departure leaves no 3-clique remaining.

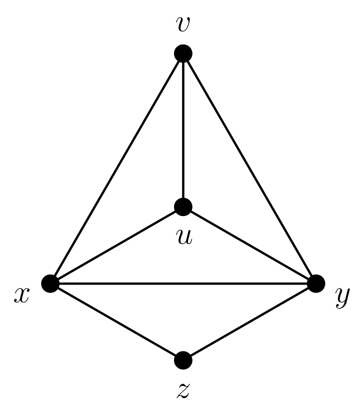
Solution. It is convenient to use the language of graph theory. Each person at the party is represented by a vertex, and there is an edge joining two vertices if the corresponding persons are acquainted. An m-clique then corresponds to a set of m vertices with each pair of vertices joined by an edge. In other words, the existence of such a clique means the given graph contains the complete graph K_m as a subgraph. In particular, a 3-clique corresponds to a triangle (K_3). We wish to prove that in any graph G in which any two triangles have at least one vertex in common and there is no K_5, there exist two or fewer vertices whose removal eliminates all triangles.

Let G be such a graph. The result is trivially true in case G has at most one triangle. Thus we have either (a) or (b) as shown below.



(a) (b)

Suppose (a) occurs, and let $T_1 = \{p, q, r\}$ and $T_2 = \{r, s, t\}$. If the deletion of r destroys all triangles, we are done. Otherwise there is a third triangle T_3 that is not destroyed by the removal of r, and this triangle must share a vertex with each of T_1 and T_2. It is plain that any such triangle leads to an occurrence of (b) with $x = r$ and $u \in T_1$, $v \in T_2$. Thus we are left to consider case (b). Suppose (b) occurs, and now let $T_1 = \{u, v, x\}$ and $T_2 = \{u, v, y\}$. If the deletion of u and v destroys all triangles, we are done. Otherwise, for some $z \notin \{u, v, x, y\}$ there must be a triangle $T_3 = \{x, y, z\}$. In particular, xy is an edge. Now G contains the following subgraph.



(c)

We claim that the deletion of x and y destroys all triangles. Suppose not. Then there is a triangle T that is disjoint from $\{x, y\}$. Since T shares a vertex with $\{x, y, z\}$, T contains z. Similarly, T contains u since it shares a vertex with $\{x, y, u\}$ and T contains v since it shares a vertex with $\{x, y, v\}$. Thus $T = \{z, u, v\}$, but this is impossible since G contains no K_5. Hence there are always two or fewer vertices whose removal destroys all triangles.

Problem C4. A set of three nonnegative integers $\{x, y, z\}$ with $x < y < z$ is called *historic* if $\{z-y, y-x\} = \{1776, 2001\}$. Show that the set of all nonnegative integers can be written as the union of pairwise disjoint historic sets.

Solution. For convenience let $a = 1776$ and $b = 2001$. All that we will really use about a and b is that $0 < a < b$. Define

$$A = \{0, a, a+b\}$$
$$B = \{0, b, a+b\}.$$

Note that both A and B are historic, and that a set X is historic if and only if $X = x+A$ or $X = x+B$ for some nonnegative integer x, where $x+S = \{x+s | s \in S\}$.

We will show how to construct an infinite sequence $X_0, X_1, X_2, \ldots$ of disjoint historic sets with the property that if k is the smallest nonnegative integer not included among X_0 through X_m, then k belongs to X_{m+1}. Thus the union of this infinite sequence includes every nonnegative integer.

Take $X_0 = A$. Assuming that we have constructed X_0 through X_m, let k be the least element not occurring in their union, U. Then take $X_{m+1} = k + A$ if $k + a \notin U$ and $k + B$ otherwise. That is, always take $k + A$ first, if possible.

Why does this construction never fail? Suppose that we had carried it out to some point m, and then failed. Note that the smallest elements of X_0 through X_m are all less than k (since at each stage we added a set whose smallest element was the first missing from the union of the earlier ones). Therefore the element $k + a + b$ is not in U. So the failure must be due to the fact that $k + b$ is covered by U. How was $k + b$ covered? For some $j \leq m$, it must have been the largest element of X_j. Let l denote the least element in X_j. Then $k + b = l + a + b$, so $k = l + a$. Since k is not covered, $X_j = l + B$. But by the algorithm, we cannot choose $X_j = l + B$ when $l + a$ is not covered, a contradiction. This contradiction shows that the construction can never fail.

Problem C5. Find all finite sequences $(x_0, x_1, \ldots, x_n)$ such that for every j, $0 \le j \le n$, x_j equals the number of times j appears in the sequence.

Solution. Let $(x_0, x_1, \ldots, x_n)$ be any such sequence. Since each x_j is the number of times j appears, the terms of the sequence are nonnegative integers. Note that $x_0 > 0$ since $x_0 = 0$ is a contradiction. Let m denote the number of positive terms among $x_1, x_2, \ldots, x_n$. Since $x_0 = p \ge 1$ implies $x_p \ge 1$, we see that $m \ge 1$. Observe that $\sum_{i=1}^{n} x_i = m + 1$ since the sum on the left counts the total number of positive terms of the sequence, and $x_0 > 0$. (*Note.* For every $j > 0$ that appears as some x_i, the sequence is long enough to include a term x_j to count it, because the sequence contains j values of i and at least one other value, the value j itself if $i \ne j$ and the value 0 if $i = j$.) Since the sum has exactly m positive terms, $m-1$ of its terms equal 1, one term equals 2, and the remainder are 0. Therefore only x_0 can exceed 2, so for $j > 2$ the possibility that $x_j > 0$ arises only in case $j = x_0$. In particular, $m \le 3$. Hence there are three cases to consider. In each case, bear in mind that $m - 1$ of the terms $x_1, x_2, \ldots, x_n$ equal 1, one term equals 2, and the the others are 0.

Case (i): $m = 1$. We have $x_2 = 2$ since $x_1 = 2$ is impossible. Thus $x_0 = 2$ and the final sequence is $(2, 0, 2, 0)$.

Case (ii): $m = 2$. Either $x_1 = 2$ or $x_2 = 2$. The first possibility leads to $(1, 2, 1, 0)$ and the second one gives $(2, 1, 2, 0, 0)$.

Case (iii): $m = 3$. In this case, $x_p > 0$ for some $p \ge 3$. By the last sentence before Case (i), $x_0 = p$ and $x_p = 1$. Then $x_1 = 1$ is contradictory, so $x_1 = 2$, $x_2 = 1$, and we have accounted for all of the positive terms of the sequence. The resulting sequence is $(p, 2, 1, \underbrace{0, \ldots, 0}_{p-3}, 1, 0, 0, 0)$.

In summary, there are three special solutions and one infinite family:

$$(2,0,2,0), \quad (1,2,1,0), \quad (2,1,2,0,0), \quad (p,2,1,\underbrace{0,\ldots,0}_{p-3},1,0,0,0), \ \ p \ge 3.$$

Note. If one considers the null set to be a sequence, then it too is a solution.

Comment. An expanded version of the problem allows for infinite sequences, and such solutions exist. One simple construction starts with a finite solution $(x_0, x_1, \ldots, x_n)$, sets $x_{n+1} = n + 1$ and continues as shown:

$$(x_0, x_1, \ldots, x_n, \underbrace{n+1, n+1, \ldots, n+1}_{x_{n+1}=n+1 \text{ terms}}, \underbrace{n+2, n+2, \ldots, n+2}_{x_{n+2} \text{ terms}}, \ldots).$$

For example, $(1, 2, 1, 0, 4, 4, 4, 4, 5, 5, 5, 5, 6, 6, 6, 6, 7, 7, 7, 7, 8, 8, 8, 8, 8, \ldots)$.

Problem C6. For a positive integer n define a sequence of zeros and ones to be *balanced* if it contains n zeros and n ones. Two balanced sequences a and b are *neighbors* if you can move one of the $2n$ symbols of a to another position to form b. For instance, when $n = 4$, the balanced sequences 01101001 and 00110101 are neighbors because the third (or fourth) zero in the first sequence can be moved to the first or second position to form the second sequence. Prove that there is a set S of at most $\frac{1}{n+1}\binom{2n}{n}$ balanced sequences such that every balanced sequence is equal to or is a neighbor of at least one sequence in S.

Solution. For each balanced sequence $a = (a_1, a_2, \ldots, a_{2n})$ let $f(a)$ be the sum of the positions of the 1's in a. For example, $f(01101001) = 2+3+5+8 = 18$. Partition the $\binom{2n}{n}$ balanced sequences into $n+1$ classes according to the residue of $f \pmod{n+1}$, and let S be a class of minimum size. Then $|S| \leq \frac{1}{n+1}\binom{2n}{n}$, and we claim that every balanced sequence is either a member of S or is a neighbor of at least one member of S. Let $a = (a_1, a_2, \ldots, a_{2n})$ be a given balanced sequence. We consider two cases.

Case (i): $a_1 = 1$. The balanced sequence $b = (b_1, b_2, \ldots, b_{2n})$ obtained from a by moving the leftmost 1 just to the right of the kth 0 satisfies $f(b) = f(a) + k$. (If a_{m+1} is the kth 0 of a, then in going from a to b, the leftmost 1 is moved up m places and $m - k$ 1's are moved back one place each.) Thus we find n neighbors of a so that the values of f for a and these neighbors fill an interval of $n + 1$ consecutive integers. In particular, one of these $n + 1$ balanced sequences belongs to S.

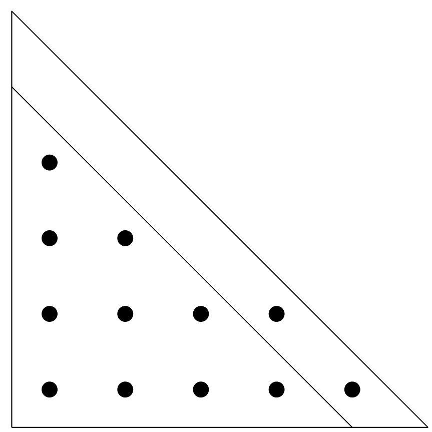
Case (ii): $a_1 = 0$. This case is similar. Movement of the initial 0 just to the right of the kth 1 yields a neighbor b satisfying $f(b) = f(a) - k$.

Hence every balanced sequence is either equal to or is a neighbor of at least one member of S.

Problem C7. A pile of n pebbles is placed in a vertical column. This configuration is modified according to the following rules. A pebble can be moved if it is at the top of a column which contains at least two more pebbles than the column immediately to its right. (If there are no pebbles to the right, think of this as a column with 0 pebbles.) At each stage, choose a pebble from among those that can be moved (if there are any) and place it at the top of the column to its right. If no pebbles can be moved, the configuration is called a *final configuration*. For each n, show that, no matter what choices are made at each stage, the final configuration obtained is unique. Describe that configuration in terms of n.

Alternative Version. A pile of 2001 pebbles is placed in a vertical column. This configuration is modified according to the following rules. A pebble can be moved if it is at the top of a column which contains at least two more pebbles than the column immediately to its right. (If there are no pebbles to the right, think of this as a column with 0 pebbles.) At each stage, choose a pebble from among those that can be moved (if there are any) and place it at the top of the column to its right. If no pebbles can be moved, the configuration is called a *final configuration*. Show that, no matter what choices are made at each stage, the final configuration obtained is unique. Describe that configuration as follows: Determine the number, c, of nonempty columns, and for each $i = 1, 2, \ldots, c$, determine the number of pebbles p_i in column i, where column 1 is the leftmost column, column 2 the next to the right, and so on.

Solution 1 of the First Version. At any stage, let p_i be the number of pebbles in column i for $i = 1, 2, \ldots$, where column 1 denotes the leftmost column. We will show that in the final configuration, for all i for which $p_i > 0$ we have $p_i = p_{i+1} + 1$, except that for at most one i^*, $p_{i^*} = p_{i^*+1}$. Therefore, the configuration looks like the figure shown below, where there are c nonempty columns and there are from 1 to c pebbles in the last diagonal row of the triangular configuration. In particular, let $t_k = 1 + 2 + \cdots + k = k(k+1)/2$ be the kth triangular number. Then c is the unique integer for which $t_{c-1} < n \le t_c$. Let $s = n - t_{c-1}$. Then there are s pebbles in the rightmost diagonal, and so the two columns with the same height are columns $c - s$ and $c{-}s{+}1$ (except if $s = c$, in which case no nonempty columns have equal height).



Final Configuration for $n = 12$

Another way to say this is

$$p_i = \begin{cases} c - i & \text{if } i \le c - s, \\ c - i + 1 & \text{if } i > c - s. \end{cases} \tag{1}$$

To prove this claim, we show that

(a) At any stage of the process, $p_1 \ge p_2 \ge \cdots$

(b) At any stage, it is not possible for there to be $i < j$ for which $p_i = p_{i+1}$, $p_j = p_{j+1} > 0$, and $p_{i+1} - p_j \le j - i - 1$ (that is, the average decrease per column from column $i + 1$ to column j is 1 or less).

(c) At any final configuration, $p_i - p_{i+1} = 0$ or 1, with at most one i for which $p_i > 0$ and $p_i - p_{i+1} = 0$.

In the proofs of (a)-(c), we use the following terminology. Let a *k-switch* be the movement of one pebble from column k to column $k + 1$, and for any column i let a *drop* be the quantity $p_i - p_{i+1}$.

To prove (a), suppose a sequence of valid moves resulted in $p_i < p_{i+1}$ for the first time at some stage j. Then the move leading to this stage must have been an i-switch, but it would be contrary to the condition that column i have at least 2 more pebbles than column $i+1$, to allow switches.

To prove (b), if such a configuration were obtainable, there would be a minimum value of $j - i$ over all such obtainable configurations, and we now show that there is no minimum. Suppose $p_1, p_2, \ldots$ was such a minimal configuration. It cannot be that $j = i + 1$, for what would columns $i, i+1, i+2$ look like just before the move that made the heights equal? The move must have been a k-switch for $i - 1 \leq k \leq i + 2$, but if so the configuration before the switch was impossible (not decreasing).

Now suppose $j > i + 1$. Consider the first configuration C in the sequence for which columns $i, i+1, j, j+1$ are at their final heights. Note that from p_{i+1} to p_j the columns decrease by exactly one each time in C, because if there was a drop of 2 or more at some point, there would have to be another drop of 0 in this interval to obtain an average of 1 or less, and thus $j - i$ is not minimal. The move leading to C was either an i-switch or a j-switch. If it was the former, at the previous stage columns $i + 1$ and $i + 2$ had the same height, violating the minimality of $j - i$. A similar contradiction arises if the move was a j-switch.

Finally, to prove (c), if any drop is 2 or more, the configuration isn't final. However, if all drops are 0 or 1, and there were two drops of 0 between nonempty columns (say between i and $i+1$ and between j and $j+1$), then (b) would be violated. Thus a final configuration that satisfied (b) also satisfies (c).

Solution of the Alternative Version. Same as above, except after display (1) insert:

Direct calculation shows that $2001 = t_{63} - 15$, so there are 63 nonempty columns and the final configuration is

$$p_i = \begin{cases} 63 - i & \text{if } i \leq 15, \\ 64 - i & \text{if } 16 \leq i \leq 63. \end{cases}$$

Solution 2 of the First Version. At each stage, let c be the rightmost nonempty column. In conditions (a)-(c) in the previous solution, replace (b) by (b′), where

(b′) All configurations obtainable from the initial configuration satisfy

$$p_i - p_j \geq j - i - 1 \quad \text{for all } i < j \leq c + 1. \tag{2}$$

(The restriction to $j \leq c + 1$, which causes certain complications, is necessary for (2) to be true.) Fact (c), and thus the answer, follows as easily from (b′) as from (b). We prove (b′) by induction as follows.

Condition (2) is immediate for the initial configuration: Since $c = 1$, the only case is $p_1 - p_2 = n > 2{-}1{-}1$. Now suppose some configuration $p_1, p_2 \ldots$ with final nonempty column c_p satisfies (2), and a new configuration $q_1, q_2 \ldots$ is obtained from it by a k-switch. Thus $q_k = p_k - 1$, $q_{k+1} = p_{k+1} + 1$, and $q_i = p_i$ for all other i. Let the new configuration have c_q nonempty columns. Note that $c_q = c_p$ unless $k = c_p - 1$.

For any $i < j \leq c_q + 1$ we now show that $q_i - q_j \geq j - i - 1$. The only cases to consider are those where $q_i - q_j < p_i - p_j$, that is, those where $i = k$ or $j = k + 1$; and those where $p_i - p_j$ wasn't restricted, because j was greater than $c_p + 1$ (case 4 below). There are four such cases.

Case 1. If $(i, j) = (k, k{+}1)$, then $q_i - q_j \geq 0 = j - i - 1$.

Case 2. If $i = k$ and $j > k + 1$, apply (2) to $(i{+}1, j)$ to obtain

$$q_i - q_j \geq q_{i+1} - q_j = p_{i+1} - p_j + 1 \geq j - (i{+}1) - 1 + 1 = j - i - 1.$$

Case 3. If $j = k + 1$ and $i < k$, then applying (2) to $(i, j{-}1)$,

$$q_i - q_j \geq q_i - q_{j-1} = p_i - p_{j-1} + 1 \geq (j{-}1) - i - 1 + 1 = j - i - 1.$$

Case 4. We have $j = c_p + 2 = k{+}2$, $p_{k+1} = 0$ and $p_k \geq 2$. If $i = k$ or $k{+}1$, then $q_i - q_j = q_i \geq 1 \geq j{-}i{-}1$. If $i < k$, then

$$q_i - q_j = p_i - 0 \geq p_i - p_k + 2 \geq (i{-}k{-}1) + 2 = i{-}j - 1.$$

This concludes the inductive step and (b′) is proved.

Problem C8. Twenty-one girls and twenty-one boys took part in a mathematical competition. It turned out that

(a) each contestant solved at most six problems, and

(b) for each pair of a girl and a boy, there was at least one problem that was solved by both the girl and the boy.

Show that there is a problem that was solved by at least three girls and at least three boys.

Solution 1. We introduce the following symbols: G is the set of girls at the competition and B is the set of boys, P is the set of problems, $P(g)$ is the set of problems solved by $g \in G$, and $P(b)$ is the set of problems solved by $b \in B$. Finally, $G(p)$ is the set of girls that solve $p \in P$ and $B(p)$ is the set of boys that solve p. In terms of this notation, we have that for all $g \in G$ and $b \in B$,

$$\text{(a)} \;\; |P(g)| \leq 6, \; |P(b)| \leq 6, \qquad \text{(b)} \;\; P(g) \cap P(b) \neq \varnothing.$$

We wish to prove that some $p \in P$ satisfies $|G(p)| \geq 3$ and $|B(p)| \geq 3$. To do this, we shall assume the contrary and reach a contradiction by counting (two ways) all ordered triples (p, q, r) such that $p \in P(g) \cap P(b)$. With $T = \{(p, g, b) : \;\; p \in P(g) \cap P(b)\}$, condition (b) yields

$$|T| = \sum_{g \in G} \sum_{b \in B} |P(g) \cap P(b)| \geq |G| \cdot |B| = 21^2. \tag{1}$$

Assume that no $p \in P$ satisfies $|G(p)| \geq 3$ and $|B(p)| \geq 3$. We begin by noting that

$$\sum_{p \in P} |G(p)| = \sum_{g \in G} |P(g)| \leq 6|G| \qquad \text{and} \qquad \sum_{p \in P} |B(p)| \leq 6|B|. \tag{2}$$

(*Note.* The equality in (2) is obtained by a standard double-counting technique: Let $\chi(g, p) = 1$ if g solves p and $\chi(g, p) = 0$ otherwise, and interchange the orders of summation in $\sum_{p\in P}\sum_{g\in G}\chi(g,p)$.) Let

$$\begin{aligned} P_+ &= \{p \in P: \ |G(p)| \geq 3\}, \\ P_- &= \{p \in P: \ |G(p)| \leq 2\}. \end{aligned}$$

Claim. $\sum_{p\in P_-}|G(p)| \geq |G|$; thus $\sum_{p\in P_+}|G(p)| \leq 5|G|$. Also $\sum_{p\in P_+}|B(p)| \geq |B|$; thus $\sum_{p\in P_-}|B(b)| \leq 5|B|$.

Proof. Let $g \in G$ be arbitrary. By the Pigeonhole Principle, conditions (a) and (b) imply that g solves some problem p that is solved by at least $\lceil 21/6 \rceil = 4$ boys. By assumption, $|B(p)| \geq 4$ implies that $p \in P_-$, so every girl solves at least one problem in P_-. Thus

$$\sum_{p\in P_-}|G(p)| \geq |G|. \tag{3}$$

In view of (2) and (3) we have

$$\sum_{p\in P_+}|G(p)| = \sum_{p\in P}|G(p)| - \sum_{p\in P_-}|G(p)| \leq 5|G|.$$

Also, each boy solves a problem that is solved by at least four girls, so each boy solves a problem $p \in P_+$. Thus $\sum_{p\in P_+}|B(p)| \geq |B|$, and the calculation proceeds as before using (2). $\square$

Using the claim just established, we find

$$\begin{aligned} |T| &= \sum_{p\in P}|G(p)|\cdot|B(p)| \\ &= \sum_{p\in P_+}|G(p)|\cdot|B(p)| + \sum_{p\in P_-}|G(p)|\cdot|B(p)| \\ &\leq 2\sum_{p\in P_+}|G(p)| + 2\sum_{p\in P_-}|B(p)| \\ &\leq 10|G| + 10|B| = 20\cdot 21. \end{aligned}$$

This contradicts (1), so the proof is complete.

Solution 2. Let us use some of the notation given in the first solution. Suppose that for every $p \in P$ either $|G(p)| \leq 2$ or $|B(p)| \leq 2$. For each $p \in P$, color p red if $|G(p)| \leq 2$ and otherwise color it black. In this way, if p is red then $|G(p)| \leq 2$ and if p is black then $|B(p)| \leq 2$. Consider a chessboard with 21 rows, each representing one of the girls, and 21 columns, each representing one of the boys. For each $g \in G$ and $b \in B$, color the square corresponding to (g, b) as follows: pick $p \in P(g) \cap P(b)$ and assign p's color to that square. (By condition (b), there is always an available choice.) By the Pigeonhole Principle, one of the two colors is assigned to at least $\lceil 441/2 \rceil = 221$ squares, and thus some row has at least $\lceil 221/21 \rceil = 11$ black squares or some column has at least 11 red squares.

Suppose the row corresponding to $g \in G$ has at least 11 black squares. Then for each of 11 squares, the black problem that was chosen in assigning the color was solved by at most 2 boys. Thus we account for at least $\lceil 11/2 \rceil = 6$ distinct problems solved by g. In view of condition (a), g solves only these problems. But then at most 12 boys solve a problem also solved by g, in violation of condition (b).

In exactly the same way, a contradiction is reached if we suppose that some column has at least 11 red squares. Hence some $p \in P$ satisfies $|G(p)| \geq 3$ and $|B(p)| \geq 3$.

Chapter 4

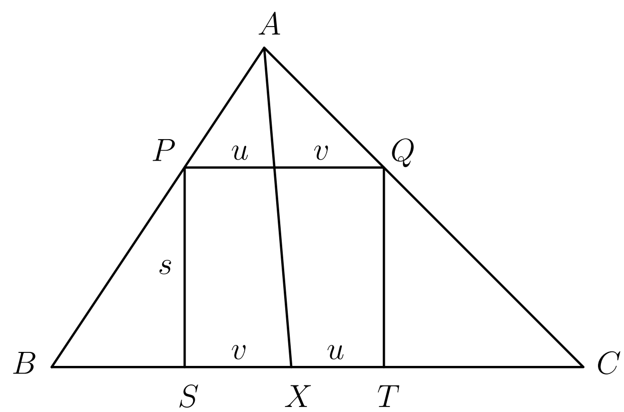
Geometry

Problem G1. Let A_1 be the center of the square inscribed in acute triangle ABC with two vertices of the square on side BC. Thus one of the two remaining vertices of the square is on side AB and the other is on AC. Points $B_1,\ C_1$ are defined in a similar way for inscribed squares with two vertices on sides AC and AB, respectively. Prove that lines $AA_1,\ BB_1,\ CC_1$ are concurrent.

Solution. Let $\alpha = \angle CAB$, $\beta = \angle ABC$, and $\gamma = \angle BCA$ be the angles of triangle ABC. Let the line through A and A_1 meet side BC at X. Similarly, let the line through B and B_1 meet side CA at Y, and the line through C and C_1 meet side AB at Z. By the converse of Ceva's Theorem, it suffices to prove that

$$\frac{BX}{XC}\frac{CY}{YA}\frac{AZ}{ZB} = 1.$$

Consider first BX/XC. Let the square with center A_1 have side s, vertices P and Q on sides AB and AC, respectively, and vertices S and T on BC with S between B and T. Since AX passes through the center of the square $QPST$, if it cuts side PQ of the square into segments of length u and v, then it cuts side ST into segments of length v and u as shown.



We then have

$$\frac{BX}{XC} = \frac{u}{v} = \frac{BX+u}{XC+v} = \frac{BT}{SC} = \frac{BS+s}{TC+s} = \frac{s\cot\beta+s}{s\cot\gamma+s} = \frac{\cot\beta+1}{\cot\gamma+1}.$$

Similarly,

$$\frac{CY}{YA} = \frac{\cot\gamma+1}{\cot\alpha+1} \qquad \text{and} \qquad \frac{AZ}{ZB} = \frac{\cot\alpha+1}{\cot\beta+1}.$$

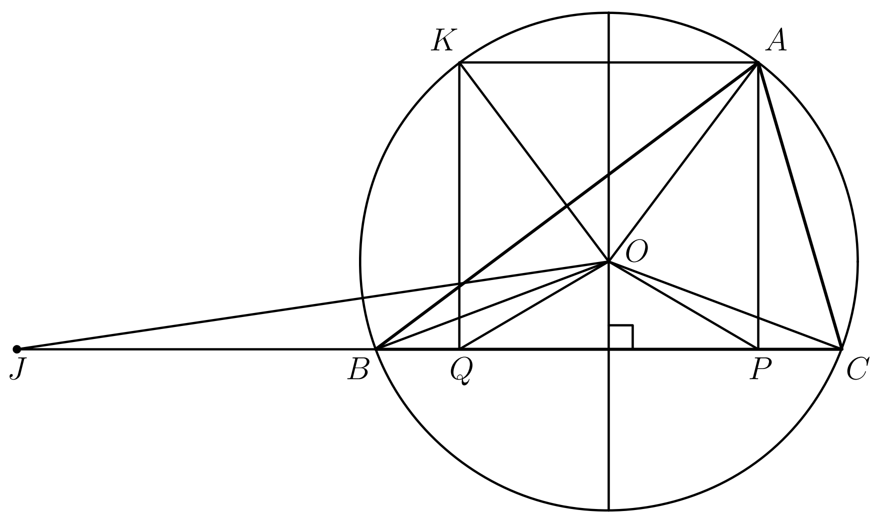
Hence

$$\frac{BX}{XC}\,\frac{CY}{YA}\,\frac{AZ}{ZB} = 1,$$

completing the proof.

Problem G2. In acute triangle ABC with circumcenter O and altitude AP, $\angle C \geq \angle B + 30^\circ$. Prove that $\angle A + \angle COP < 90^\circ$.

Solution 1. Let $\alpha = \angle CAB$, $\beta = \angle ABC$, $\gamma = \angle BCA$, and $\delta = \angle COP$. Let K and Q be the reflections of A and P, respectively, across the perpendicular bisector of BC. Let R denote the circumradius of $\triangle ABC$. Then $OA = OB = OC = OK = R$. Furthermore, we have $QP = KA$ because $KQPA$ is a rectangle. Now note that $\angle AOK = \angle AOB - \angle KOB = \angle AOB - \angle AOC = 2\gamma - 2\beta \geq 60^\circ$.



It follows from this and from $OA = OK = R$ that $KA \geq R$ and $QP \geq R$. Therefore, using the Triangle Inequality, we have $OP + R = OQ + OC > QC = QP + PC \geq R + PC$. It follows that $OP > PC$, and hence in $\triangle COP$, $\angle PCO > \delta$. Now since $\alpha = \frac{1}{2}\angle BOC = \frac{1}{2}(180^\circ - 2\angle PCO) = 90^\circ - \angle PCO$, it indeed follows that $\alpha + \delta < 90^\circ$.

Solution 2. As in the previous solution, it is enough to show that $OP > PC$. To this end, recall that by the (Extended) Law of Sines, $AB = 2R\sin\gamma$ and $AC = 2R\sin\beta$. Therefore, we have

$$BP - PC = AB\cos\beta - AC\cos\gamma = 2R(\sin\gamma\cos\beta - \sin\beta\cos\gamma) = 2R\sin(\gamma - \beta).$$

It follows from this and from

$$30^\circ \le \gamma - \beta < \gamma < 90^\circ$$

that $BP - PC \ge R$. Therefore, we obtain that $R + OP = BO + OP > BP \ge R + PC$, from which $OP > OC$, as desired.

Solution 3. We first show that $R^2 > CP \cdot CB$. To this end, since $CB = 2R\sin\alpha$ and $CP = AC\cos\gamma = 2R\sin\beta\cos\gamma$, it suffices to show that $\frac{1}{4} > \sin\alpha\sin\beta\cos\gamma$. We note that $1 > \sin\alpha = \sin(\gamma + \beta) = \sin\gamma\cos\beta + \sin\beta\cos\gamma$ and $\frac{1}{2} \le \sin(\gamma - \beta) = \sin\gamma\cos\beta - \sin\beta\cos\gamma$ since $30^\circ \le \gamma - \beta < 90^\circ$. It follows that $\frac{1}{4} > \sin\beta\cos\gamma$ and that $\frac{1}{4} > \sin\alpha\sin\beta\cos\gamma$.

Now we choose a point J on BC so that $CJ \cdot CP = R^2$. It follows from this and from $R^2 > CP \cdot CB$ that $CJ > CB$, so that $\angle OBC > \angle OJC$. Since $OC/CJ = PC/CO$ and $\angle JCO = \angle OCP$, we have $\triangle JCO \cong \triangle OCP$ and $\angle OJC = \angle POC = \delta$. It follows that $\delta < \angle OBC = 90^\circ - \alpha$ or $\alpha + \delta < 90^\circ$.

Solution 4. On the one hand, as in the third solution, we have $R^2 > CP \cdot CB$. On the other hand, the power of P with respect to the circumcircle of $\triangle ABC$ is $BP \cdot PC = R^2 - OP^2$. From these two equations we find that

$$OP^2 = R^2 - BP \cdot PC > PC \cdot CB - BP \cdot PC = PC^2,$$

from which $OP > PC$. Therefore, as in the first solution, we conclude that $\alpha + \delta < 90^\circ$.

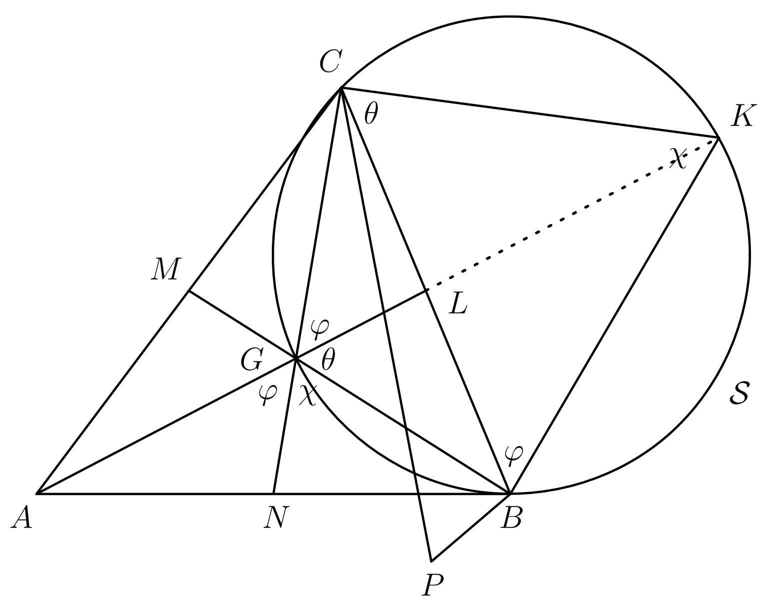
Problem G3. Let ABC be a triangle with centroid G. Determine, with proof, the position of the point P in the plane of ABC such that $AP{\cdot}AG + BP{\cdot}BG + CP{\cdot}CG$ is a minimum, and express this minimum value in terms of the side lengths of ABC.

Solution. As usual, let a, b, c denote the sides of the triangle facing the vertices A, B, C, respectively. We will show that the desired minimum value of $AP \cdot AG + BP \cdot BG + CP \cdot CG$ is attained when P is the centroid G, and that the minimum value is

$$\begin{aligned} AG^2 + BG^2 + CG^2 &= \tfrac{1}{9}\{(2b^2 + 2c^2 - a^2) + (2c^2 + 2a^2 - b^2) + (2c^2 + 2a^2 - b^2)\} \\ &= \frac{a^2 + b^2 + c^2}{3}. \end{aligned}$$

The latter follows by using Stewart's Theorem to compute the lengths of the medians AL, BM, CN, along with the relations $AG = \frac{2}{3}AL$, $BG = \frac{2}{3}BM$, $CG = \frac{2}{3}CN$.

Let $\mathcal{S}$ be the circle passing through B, G, and C. The median AL meets $\mathcal{S}$ at G and K. Let θ, ϕ, χ be the angle measures as shown.



By the Law of Sines, we find

$$\frac{BG}{CG} = \frac{\sin\varphi}{\sin\theta} \qquad \text{and} \qquad \frac{AG}{BG} = \frac{\sin\chi}{\sin\varphi}.$$

Also $BK = 2R\sin\theta$, $CK = 2R\sin\varphi$, $BC = 2R\sin\chi$, where R is the radius of $\mathcal{S}$. Hence

$$\frac{CG}{BK} = \frac{BG}{CK} = \frac{AG}{BC}. \tag{$*$}$$

Let P be any point in the plane of ABC. By Ptolemy's Theorem,

$$PK \cdot BC \leq BP \cdot CK + BK \cdot CP,$$

with equality if and only if P lies on $\mathcal{S}$. In view of ($*$), we have

$$PK \cdot AG \leq BP \cdot BG + CG \cdot CP.$$

Addition of $AP \cdot AG$ to both sides gives

$$(AP + PK) \cdot AG \leq AP \cdot AG + BP \cdot BG + CP \cdot CG.$$

Since $AK \leq AP + AK$ by the Triangle Inequality, we have

$$AK \cdot AG \leq AP \cdot AG + BP \cdot BG + CP \cdot CG.$$

Equality holds if and only if P lies on the segment AK and P lies on $\mathcal{S}$ as well. Hence equality holds if and only if $P = G$.

Problem G4. Let M be a point in the interior of triangle ABC. Let A' lie on BC with MA' perpendicular to BC. Define B' on CA and C' on AB similarly. Define

$$p(M) = \frac{MA' \cdot MB' \cdot MC'}{MA \cdot MB \cdot MC}.$$

Determine, with proof, the location of M such that $p(M)$ is maximal. Let $\mu(ABC)$ denote this maximum value. For which triangles ABC is the value of $\mu(ABC)$ maximal?

Solution. Let α, β, γ denote the angles A, B, C respectively. Also let

$$\begin{aligned} \alpha_1 &= \angle MAB, & \alpha_2 &= \angle MAC, \\ \beta_1 &= \angle MBC, & \beta_2 &= \angle MBA, \\ \gamma_1 &= \angle MCA, & \gamma_2 &= \angle MCB. \end{aligned}$$

We have

$$\frac{MB' \cdot MC'}{(MA)^2} = \sin\alpha_1 \sin\alpha_2, \quad \frac{MB' \cdot MA'}{(MC)^2} = \sin\gamma_1 \sin\gamma_2, \quad \frac{MA' \cdot MC'}{(MB)^2} = \sin\beta_1 \sin\beta_2,$$

so that $p(M)^2 = \sin\alpha_1 \sin\alpha_2 \sin\beta_1 \sin\beta_2 \sin\gamma_1 \sin\gamma_2$. Observe that

$$\sin\alpha_1 \sin\alpha_2 = \tfrac{1}{2}(\cos(\alpha_1 - \alpha_2) - \cos(\alpha_1 + \alpha_2)) \le \tfrac{1}{2}(1 - \cos\alpha) = \sin^2\frac{\alpha}{2}. \tag{1}$$

Likewise,

$$\sin\beta_1 \sin\beta_2 \le \sin^2\frac{\beta}{2} \quad \text{and} \quad \sin\gamma_1 \sin\gamma_2 \le \sin^2\frac{\gamma}{2}. \tag{2}$$

Therefore

$$p(M) \le \sin\frac{\alpha}{2} \sin\frac{\beta}{2} \sin\frac{\gamma}{2}.$$

Clearly, equality is achieved in (1) and (2) if and only if $\alpha_1 = \alpha_2, \beta_1 = \beta_2, \gamma_1 = \gamma_2$; in other words, $p(M)$ achieves its maximum value when M is the center of the inscribed circle of triangle ABC and this maximum value is

$$\mu(ABC) = \sin\frac{\alpha}{2} \sin\frac{\beta}{2} \sin\frac{\gamma}{2}.$$

It is well known that this quantity is maximal when the triangle is equilateral. This can be proven in many ways; for example, using Jensen's inequality. A more elementary proof uses the first equality of (1) to deduce that if $x, y \geq 0$ and $x + y \leq \pi/2$ is fixed, the value of $\sin x \sin y$ will increase as the difference $|x - y|$ decreases. Thus, if $x + y + z = \pi/2$, the value of $\sin x \sin y \sin z$ can be increased if any of the x, y, z are not equal to $\pi/6$. (For example, if $x < \pi/6$ and $z > \pi/6$ and x is closer to $\pi/6$ than z is, replace x by $x' = \pi/6$ and z by $z' = z - \pi/6 + x$. The sum $x' + y + z'$ remains unchanged, but the product $\sin x' \sin y \sin z'$ increases.)

Comment. The Jury may wish to consider an alternative version of this problem, which asks only the first of the two questions (i.e., only asks for the location of M). This would avoid a situation in which some students laboriously prove that $\sin x \sin y \sin z$ is maximized when $x = y = z$, while others use Jensen's inequality, and still others merely state, as the proposer did, that the result is well known.

Problem G5. Let ABC be an acute triangle. Let DAC, EAB, and FBC be isosceles triangles exterior to ABC, with $DA = DC, EA = EB$, and $FB = FC$, such that

$$\angle ADC = 2\angle BAC, \quad \angle BEA = 2\angle ABC, \quad \angle CFB = 2\angle ACB.$$

Let D' be the intersection of lines DB and EF, let E' be the intersection of EC and DF, and let F' be the intersection of FA and DE. Find, with proof, the value of the sum

$$\frac{DB}{DD'} + \frac{EC}{EE'} + \frac{FA}{FF'}.$$

Solution. Note that $\angle ADC, \angle BEA, \angle CFB < \pi$ since ABC is an acute triangle. Also,

$$\angle DAC = \frac{\pi}{2} - \frac{1}{2}\angle ADC = \frac{\pi}{2} - \angle BAC$$

and

$$\angle BAE = \frac{\pi}{2} - \frac{1}{2}\angle BEA = \frac{\pi}{2} - \angle ABC.$$

Hence

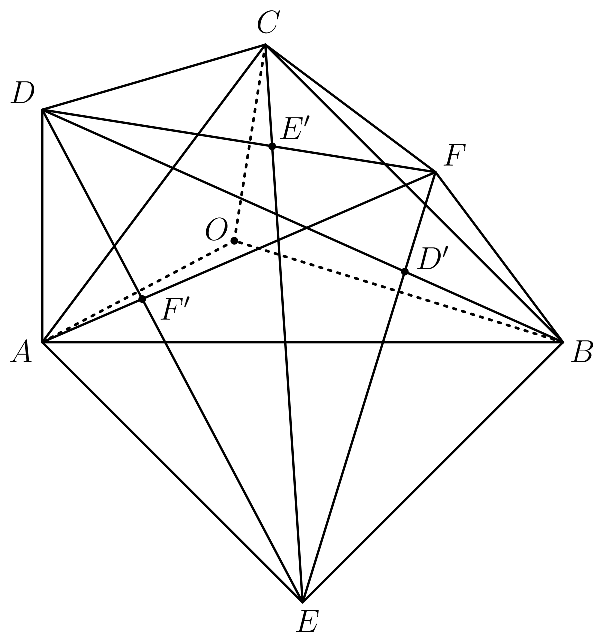
$$\angle DAE = \angle DAC + \angle BAC + \angle BAE = \pi - \angle ABC < \pi.$$

Likewise,

$$\angle EBF < \pi \quad \text{and} \quad \angle FCD < \pi.$$

Thus the polygon $DAEBFC$ is convex and

$$\angle ADC + \angle BEA + \angle CFB = 2(\angle BAC + \angle ABC + \angle ACB) = 2\pi.$$



Let $\omega_1, \omega_2, \omega_3$ be circles with centers at D, E, F, respectively, and radii DA, EB, FC, respectively. Using $\angle ADC + \angle BEA + \angle CFB = 2\pi$, it is easy to see by the Inscribed Angle Theorem that these three circles are concurrent; let O be the common point. Then O is the reflection of C with respect to DF. Likewise, O is also the reflection of A with respect to DE and the reflection of B with respect to EF. Let $[XYZ]$ denote the area of triangle XYZ. We have

$$\frac{DB}{DD'} = \frac{DD' + D'B}{DD'} = 1 + \frac{D'B}{DD'} = 1 + \frac{[EBF]}{[DEF]} = 1 + \frac{[OEF]}{[DEF]}.$$

Likewise,

$$\frac{EC}{EE'} = 1 + \frac{[ODF]}{[DEF]} \quad \text{and} \quad \frac{FA}{FF'} = 1 + \frac{[ODE]}{[DEF]}.$$

Thus

$$\frac{DB}{DD'} + \frac{EC}{EE'} + \frac{FA}{FF'} = 3 + \frac{[OEF] + [ODF] + [ODE]}{[DEF]} = 4.$$

Problem G6. Let ABC be a triangle and P an exterior point in the plane of the triangle. Suppose AP, BP, CP meet the sides BC, CA, AB (or extensions thereof) in D, E, F, respectively. Suppose further that the areas of triangles PBD, PCE, PAF are all equal. Prove that each of these areas is equal to the area of triangle ABC itself.

Solution 1. Let D, E, and F divide the sides BC, CA, and AB in the signed ratios $z/y, x/z$, and y/x, respectively. Since AD, BE, and CF are concurrent (at P), by Ceva's Theorem we may choose the ratios in this manner. Let us assume that $[ABC] = 1$, where $[UVW]$ denotes the signed area of $\triangle UVW$. Note that for P to lie outside the triangle at least one of x, y, z must be positive and at least one must be negative. Also,

$$\frac{[PBC]}{x} = \frac{[PCA]}{y} = \frac{[PAB]}{z} = \frac{[ABC]}{x+y+z}.$$

Now

$$[PBD] = \frac{[PBD]}{[PBC]} \frac{[PBC]}{[ABC]} [ABC] = \frac{z}{y+z} \frac{x}{x+y+z} = \frac{zx}{(y+z)(x+y+z)}.$$

Similarly,

$$[PCE] = \frac{xy}{(z+x)(x+y+z)} \quad \text{and} \quad [PAF] = \frac{yz}{(x+y)(x+y+z)}.$$

Since these three areas are equal, we have $y(y+z) = z(z+x) = x(x+y)$. We may assume at this stage that $z = 1$. This yields

$$y(y+1) = 1 + x = x(x+y).$$

So $x = y^2+y-1$ from the first equation, and hence we have $(y^2+y-1)^2+(y^2+y-1)y = y^2 + y$. Simplification gives $y^4 + 3y^3 - y^2 - 4y + 1 = 0$, which can be factored as

$$(y-1)(y^3 + 4y^2 + 3y - 1) = 0.$$

If $y = 1$, then we have $x = 1$, implying that P coincides with the centroid of $\triangle ABC$, contradicting the hypothesis that P lies outside the triangle.

Therefore,

$$y^3 + 4y^2 + 3y - 1 = 0.$$

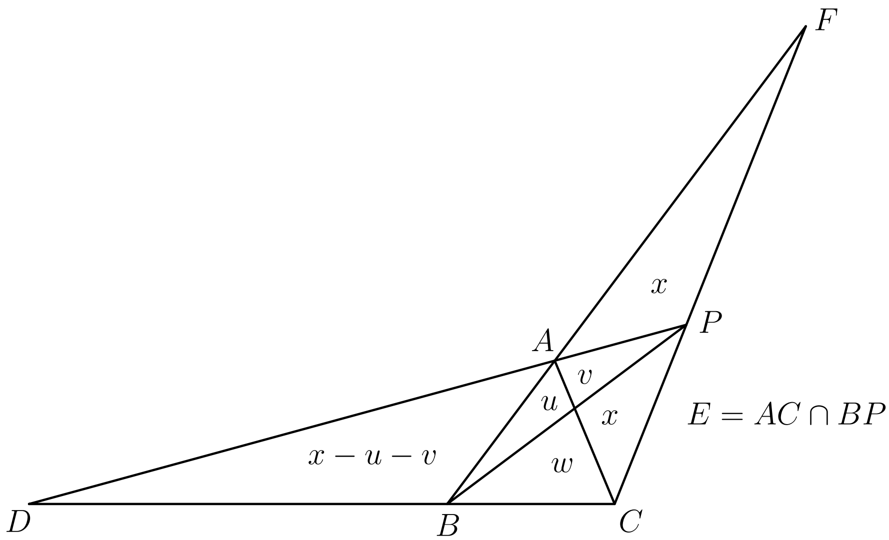
Using this fact, it follows that

$$\begin{aligned}
[PBD] &= \frac{zx}{(y+z)(x+y+z)} = \frac{x}{(1+y)(x+y+1)} \\
&= \frac{y^2+y-1}{(y+1)(y^2+2y)} = \frac{y^2+y-1}{y^3+3y^2+2y} = \frac{y^2+y-1}{-y^2-y+1} = -1, \\
[PCE] &= \frac{xy}{(z+x)(x+y+z)} = \frac{xy}{(x+1)(x+y+1)} \\
&= \frac{(y^2+y-1)y}{(y^2+y)(y^2+2y)} = \frac{y^2+y-1}{y^3+3y^2+2y} = -1, \\
[PAF] &= \frac{yz}{(x+y)(x+y+z)} = \frac{y}{(x+y)(x+y+1)} \\
&= \frac{y}{(y^2+2y-1)(y^2+2y)} = \frac{1}{y^3+4y^2+3y-2} = \frac{1}{-1} = -1.
\end{aligned}$$

These calculations also imply that not both x and y are positive. Hence P lies outside $\triangle ABC$. Moreover, $[PBD] = [PCE] = [PAF] = -1 = -[ABC]$. Hence, the desired result. The negative sign only indicates that triangles PBD, PCE, and PAF are oriented opposite to $\triangle ABC$.

Comment. Since the equation $y^3 + 4y^2 + 3y - 1 = 0$ can be solved to get three real roots in terms of $\cos\frac{2\pi}{7}, \cos\frac{4\pi}{7}$, and $\cos\frac{6\pi}{7}$, we see that there are three real positions of P lying outside $\triangle ABC$.

Solution 2. Let P be a point outside $\triangle ABC$ as shown in the diagram, and let D, E, and F be the points at which PA, PB, and PC meet the sides BC, CA, and AB, respectively. Let $[BPD] = [CPE] = [APF] = x$, $[ABE] = u, [PAE] = v$, and $[BCE] = w$, so that $[BAD] = x - u - v$. We wish to prove that $x = [ABC] = u + w$.



Now, each of the ratios, $BD/DC, CE/EA$, and AF/FB can be computed in two ways, yielding the following equations:

$$\frac{x-u-v}{x-v+w} = \frac{x}{2x+w} \tag{1}$$

$$\frac{x}{v} = \frac{w}{u} \tag{2}$$

$$\frac{x}{x+u+v} = \frac{2x+v}{2x+u+v+w} \tag{3}$$

Equation (1) gives $\frac{x}{2x+w} = \frac{u+v}{x+v}$. Simplifying and substituting for v from (2), we obtain

$$x^2(w-u) = uw(3x+w).$$

Again, simplifying (3) and using (2), we get

$$x = \frac{w(w^2 - uw - u^2)}{u(2w+u)}.$$

Eliminating x from the last two equations, we finally obtain

$$(w-u)(w+u)(w^3 - 3uw^2 - 4u^2w - u^3) = 0.$$

But $w = u$ gives $v = x$ (from (2)). So (3) gives $\frac{x}{2x+u} = \frac{3x}{3x+2u}$; i.e., $2x + u = x + \frac{2u}{3}$, which is false since the left side of this is larger than its right side. Clearly, we can also rule out $w + u = 0$. Hence, $w^3 - 3uw^2 - 4u^2w - u^3 = 0$, yielding

$$w^3 = u(3w + u)(w + u).$$

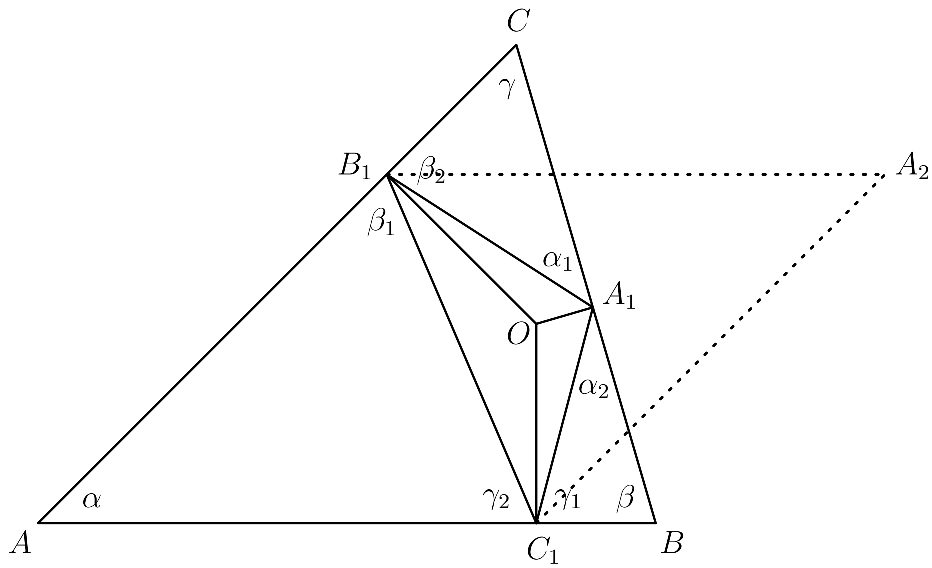
Finally, we have

$$\begin{aligned} x &= \frac{w^3 - uw^2 - u^2w}{2uw + u^2} = \frac{u(3w + u)(w + u) - uw(u + w)}{u(2w + u)} \\ &= \frac{u(u + w)(2w + u)}{u(2w + u)} = u + w, \end{aligned}$$

as desired.

Problem G7. Let O be an interior point of acute triangle ABC. Let A_1 lie on BC with OA_1 perpendicular to BC. Define B_1 on CA and C_1 on AB similarly. Prove that O is the circumcenter of ABC if and only if the perimeter of $A_1B_1C_1$ is not less than any one of the perimeters of AB_1C_1, BC_1A_1, and CA_1B_1.

Solution. If O is the circumcenter of $\triangle ABC$, then A_1, B_1, and C_1 are the midpoints of BC, CA, and AB, respectively, and hence $P_{A_1B_1C_1} = P_{AB_1C_1} = P_{BC_1A_1} = P_{CA_1B_1}$, where P_{XYZ} denotes the perimeter of $\triangle XYZ$.



Conversely, suppose that $P_{A_1B_1C_1} \geq P_{AB_1C_1}, P_{BC_1A_1}, P_{CA_1B_1}$. Let

$$\begin{array}{lll}
\angle CAB = \alpha, & \angle CA_1B_1 = \alpha_1, & \angle BA_1C_1 = \alpha_2, \\
\angle ABC = \beta, & \angle AB_1C_1 = \beta_1, & \angle CB_1A_1 = \beta_2, \\
\angle BCA = \gamma, & \angle BC_1A_1 = \gamma_1, & \angle AC_1B_1 = \gamma_2.
\end{array}$$

Let A_2 be the point of intersection of the lines through B_1 and C_1, which are parallel to AB and AC, respectively, as shown in the figure above. Assume that $\gamma_1 \geq \alpha$

and $\beta_2 \geq \alpha$. If one of these inequalities is strict, then A_1 is an interior point of $\triangle B_1C_1A_2$. Hence $P_{A_1B_1C_1} < P_{A_2B_1C_1} = P_{AB_1C_1}$, which is a contradiction. If $\gamma_1 = \alpha$ and $\beta_2 = \alpha$, then $A_1 = A_2$ and therefore $B_1O \perp A_1C_1$ and $C_1O \perp A_1B_1$. Hence O is the orthocenter (intersection of the altitudes) of $\triangle A_1B_1C_1$, and thus $OA_1 \perp B_1C_1$. Hence $B_1C_1 \parallel BC$. This implies that A_1, B_1, and C_1 are the midpoints of BC, CA, and AB, respectively; i.e., triangles $AB_1C_1, A_1B_1C_1, A_1B_1C$, and A_1BC_1 are congruent. Hence, O is the circumcenter of $\triangle ABC$. Analogously, the same conclusion holds if $\alpha_1 \geq \beta$ and $\gamma_2 \geq \beta$, or $\beta_1 \geq \gamma$ and $\alpha_2 \geq \gamma$.

Suppose now that none of these cases are satisfied; i.e., it is not true that

$$\gamma_1 \geq \alpha \quad \text{and} \quad \beta_2 \geq \alpha,$$

or

$$\alpha_1 \geq \beta \quad \text{and} \quad \gamma_2 \geq \beta,$$

or

$$\beta_1 \geq \gamma \quad \text{and} \quad \alpha_2 \geq \gamma.$$

Suppose without loss of generality that $\gamma_1 < \alpha$. Then $\alpha_2 > \gamma$, since $\gamma_1 + \alpha_2 = \pi - \beta = \alpha + \gamma$. Hence $\beta_1 < \gamma$, which implies that $\gamma_2 > \beta$. Hence $\alpha_1 < \beta$, implying that $\beta_2 > \alpha$. In conclusion,

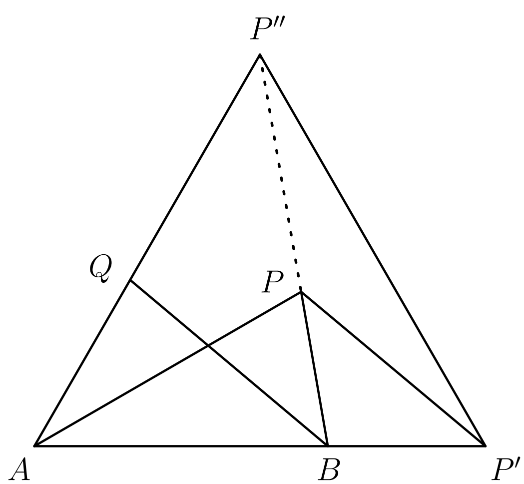
$$\gamma_1 < \alpha < \beta_2, \quad \alpha_1 < \beta < \gamma_2, \quad \text{and} \quad \beta_1 < \gamma < \alpha_2.$$

Since AC_1OB_1 and A_1CB_1O are cyclic, we have $\angle AOB_1 = \gamma_2$ and $\angle COB_1 = \alpha_1$. Hence, $AO = OB_1/\cos\gamma_2 > OB_1/\cos\alpha_1 = CO$. In the same way, the inequalities $\gamma_1 < \beta_2$ and $\beta_1 < \alpha_2$ imply that $CO > BO$ and $BO > AO$, a contradiction.

Comment. The same arguments show that O is the circumcenter of $\triangle ABC$ if and only if $P_{A_1B_1C_1} \leq P_{AB_1C_1}, P_{BC_1A_1}, P_{CA_1B_1}$.

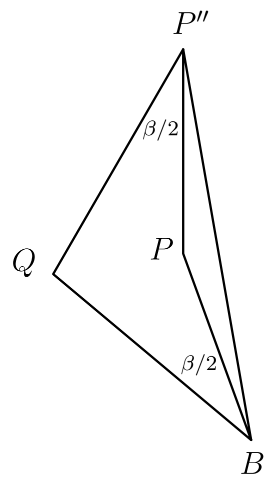
Problem G8. Let ABC be a triangle with $\angle BAC = 60°$. Let AP bisect $\angle BAC$ and let BQ bisect $\angle ABC$, with P on BC and Q on AC. If $AB + BP = AQ + QB$, what are the angles of the triangle?

Solution. Denote the angles of ABC by $\alpha = 60°$, β, and γ. Extend AB to P' so that $BP' = BP$, and construct P'' on AQ so that $AP'' = AP'$. Then $BP'P$ is an isosceles triangle with base angle $\beta/2$. Since $AQ+QP'' = AB+BP' = AB+BP = AQ+QB$, it follows that $QP'' = QB$. Since $AP'P''$ is equilateral and AP bisects the angle at A, we have $PP' = PP''$.



Claim. Points B, P, P'' are collinear, so P'' coincides with C.

Proof. Suppose to the contrary that BPP'' is a nondegenerate triangle. We have that $\angle PBQ = \angle PP'B = \angle PP''Q = \beta/2$. Thus the diagram appears as below, or else with P is on the other side of BP''. In either case, the assumption that BPP'' is nondegenerate leads to $BP = PP'' = PP'$, thus to the conclusion that BPP' is equilateral, and finally to the absurdity $\beta/2 = 60°$ so $\alpha + \beta = 60° + 120° = 180°$.



Thus points B, P, P'' are collinear, and $P'' = C$ as claimed. $\square$

Since triangle BCQ is isosceles, we have $120° - \beta = \gamma = \beta/2$, so $\beta = 80$ and $\gamma = 40°$. Thus ABC is a 60-80-40 degree triangle.

Chapter 5

Number Theory

Problem N1. Prove that there is no positive integer n such that, for $k = 1, 2, \dots, 9$, the leftmost digit (in decimal notation) of $(n+k)!$ equals k.

Solution. For each positive integer m, define

$$N(m) = \frac{m}{10^{d(m)-1}},$$

where $d(m)$ is the number of digits in m. Note that $1 \le N(m) < 10$. In addition, it is not hard to show that

$$N(lm) \le N(l)N(m) \tag{1}$$

for positive integers l and m.

Assume now that n is a positive integer such that for $k = 1, 2, \dots, 9$, the leftmost digit of $(n+k)!$ is k. If $2 \le k \le 9$, then $(n+k)! = a \times 10^r$ for some nonnegative integer r and some real number a with $k < a < k+1$, and $(n+k-1)! = b \times 10^s$ where $k-1 < b < k$ and s is a nonnegative integer. We then have

$$1 < N(n+k) = N\left(\frac{(n+k)!}{(n+k-1)!}\right) = \frac{a}{b} < \frac{k+1}{k-1} \le 3. \tag{2}$$

Now $N(m) \ge N(m+1)$ can only happen if $N(m) \ge 9$. Hence it follows from (2) that

$$1 < N(n+2) < \cdots < N(n+9) \le \tfrac{5}{4}.$$

Using (1) we then have

$$N((n+2)!) \le N((n+1)!)N(n+2) < 2 \cdot \tfrac{5}{4},$$
$$N((n+3)!) \le N((n+2)!)N(n+3) < 2(\tfrac{5}{4})^2,$$
$$N((n+4)!) \le N((n+3)!)N(n+4) < 2(\tfrac{5}{4})^3 < 4,$$

contradicting the assumption that $(n+4)!$ has leftmost digit of 4.

Therefore there is no positive integer n such that for $k = 1, 2, \ldots, 9$, the leftmost digit of $(n+k)!$ is k.

Problem N2. Consider the system

$$x + y = z + u$$
$$2xy = zu.$$

Find the greatest value of the real constant m such that $m \leq x/y$ for any positive integer solution (x, y, z, u) of the system, with $x \geq y$.

Solution 1. Squaring the first equation and then subtracting four times the second, we obtain

$$x^2 - 6xy + y^2 = (z-u)^2,$$

from which

$$\left(\frac{x}{y}\right)^2 - 6\left(\frac{x}{y}\right) + 1 = \left(\frac{z-u}{y}\right)^2. \qquad (*)$$

The quadratic $\omega^2 - 6\omega + 1$ takes the value 0 for $\omega = 3 \pm 2\sqrt{2}$, and is positive for $\omega > 3 + 2\sqrt{2}$. Because $x/y \geq 1$ and the right side of $(*)$ is a square, the left side of $(*)$ is positive, and we must have $x/y > 3 + 2\sqrt{2}$. We now show that x/y can be made as close to $3 + 2\sqrt{2}$ as we like, so the desired $m = 3 + 2\sqrt{2}$. We prove this by showing that the term $((z-u)/y)^2$ in $(*)$ can be made as small as we like.

To this end, we first find a way to generate solutions of the system. If p is a prime divisor of z and u, then p is a divisor of both x and y. Thus we may assume, without loss of generality, that z and u are relatively prime. If we square both sides of the first equation, then subtract twice the second equation we have

$$(x-y)^2 = z^2 + u^2.$$

Thus $(z,\ u,\ x-y)$ is a primitive Pythagorean triple, and we may assume that u is even. Hence there are relatively prime positive integers a and b, one of them even and the other odd, such that

$$z = a^2 - b^2, \qquad u = 2ab, \qquad \text{and} \qquad x - y = a^2 + b^2.$$

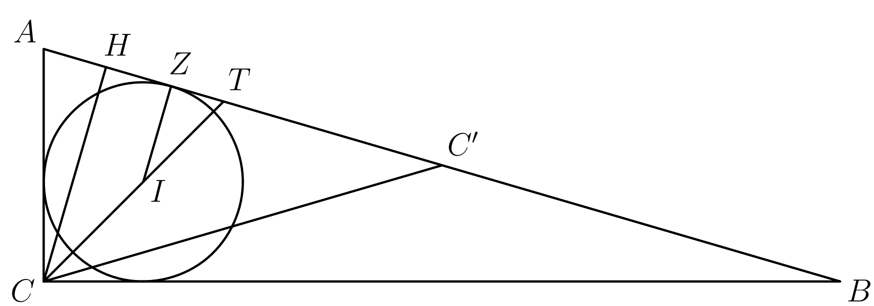
Combining these equations with $x + y = z + u$, we find that

$$x = a^2 + ab \qquad \text{and} \qquad y = ab - b^2.$$

Observe that $z - u = a^2 - b^2 - 2ab = (a-b)^2 - 2b^2$. When $z - u = 1$, we get the Pell equation $1 = (a-b)^2 - 2b^2$, which has solution $a - b = 3$, $b = 2$. By well known facts, this equation has infinitely many positive integer solutions $a - b$ and b, and both of these quantities can be made arbitrarily large. It follows that $y = ab - b^2$ can be made arbitrarily large. Hence the right side of $(*)$ can be made as small as we like, and the corresponding value of x/y can be made as close to $3 + 2\sqrt{2}$ as we like.

Note. This solution can be shortened somewhat by using a different method for generating solutions. Note that if (t, y) satisfies the Pell equation $t^2 - 2y^2 = 1$ and we set $x = 3y + 2t, z = 2y + t + 1, u = 2y + t - 1$, then $x + y = 4y + 2t = z + u$ and $2xy = 2(3y + 2t)y = 6y^2 + 4ty = (2y + t)^2 - 1 = zu$. It follows as before that there are solutions with $x/y = 3 + 2t/y$ as close to $3 + 2\sqrt{2}$ as desired.

Solution 2. As in the first solution, we find that $(x - y)^2 = u^2 + z^2$. Hence there is a right triangle with sides $a = u$ and $b = z$ and hypotenuse $c = x - y$. Let ABC be such a triangle with $AB = c$, $AC = b$, and $BC = a$. Let I be the incenter of the triangle, let r be the inradius, and let Z be the point at which the incircle is tangent to AB. Let CT be the bisector of angle C, with T on AB, let CH be the altitude to AB, and let C' be the midpoint of AB.



Because triangle ABC is right, $r = IZ = s - c$, where $s = (a+b+c)/2$ is the semiperimeter of the triangle. Thus, $a + b = 2r + c = 2r + x - y$. Using this with $a + b = u + z = x + y$ we obtain $y = r$ and $x = s$. We now prove that for any values of a and b,

$$\frac{x}{y} = \frac{s}{r} \geq (\sqrt{2}+1)^2. \tag{1}$$

To show this, observe that $CC' \geq CT \geq CI + IZ$, so

$$\frac{s-r}{2} = \frac{c}{2} \geq (\sqrt{2}+1)r.$$

It follows that

$$\frac{s}{r} \geq 2\sqrt{2} + 3 = (\sqrt{2}+1)^2,$$

which establishes (1). Equality holds only if the triangle is isosceles, but in that case the sides cannot all be of integral length. Thus the inequality in (1) is strict. On the other hand, $CH \leq CI + IZ$ so $2rs/c \leq (\sqrt{2}+1)r$. Hence

$$\frac{x}{y} = \frac{s}{r} \leq (\sqrt{2}+1)^2 \left(\frac{c^2}{4sr}\right). \tag{2}$$

However,

$$\frac{c^2}{4sr} = \frac{a^2+b^2}{2ab} = 1 + \frac{(a-b)^2}{2ab}. \tag{3}$$

Because there are infinitely Pythagorean triples (a, b, c) with $a - b = 1$, it follows from (3) that $c^2/(4rs)$ can be made as close to 1 as we like. It then follows from (1) and (2) that the maximal value for m is $3 + 2\sqrt{2}$.

Solution 3 (Sketch). Solve the first equation for u, substitute in the second, then divide by y^2 to get

$$\left(\frac{z}{y}\right)^2 - \left(\frac{z}{y}\right)\left(\frac{x}{y}\right) - \left(\frac{z}{y}\right) + 2\left(\frac{x}{y}\right) = 0.$$

Let $X = x/y$ and $Z = z/y$ to obtain

$$Z^2 - ZX - Z + 2X = 0. \tag{$*$}$$

Equation $(*)$ describes a hyperbola in the (X, Z)-plane. The asymptotes of the hyperbola have slope 0 and 1. One branch of the hyperbola lies in the half plane $X \geq 1$, the other in the half plane $X < 1$. Furthermore, the leftmost point on the branch in the half plane $X \geq 1$ has coordinates $(3 + 2\sqrt{2}, 2 + \sqrt{2})$. Thus, if (X, Z) is on the hyperbola and $X \geq 1$, then $X \geq 3 + 2\sqrt{2}$, and this bound cannot be improved. Because $(*)$ has rational coefficients and the hyperbola described by $(*)$ has a point with rational coordinates, it has infinitely many points with rational coordinates. In particular, let r be a rational number, $r \neq 0$ or 1. Then the line with equation $y = rx$ intersects the hyperbola described by $(*)$ in the points $(0, 0)$ and $\left(\frac{r-2}{r^2-r}, \frac{r^2-2r}{r^2-r}\right)$. If $0 < r < 1$, then the latter point is on the right branch of the hyperbola, so has (rational) X coordinate $\frac{r-2}{r^2-r} > 3 + 2\sqrt{2}$. Furthermore, if r is close to

$$\frac{2+\sqrt{2}}{3+2\sqrt{2}} = 2 - \sqrt{2},$$

then this X-coordinate is close to $3+2\sqrt{2}$. Finally, if X and Z are positive rationals satisfying $(*)$, then we can find integers x, y, and z with $X = x/y$ and $Z = z/y$. It is easy to check that x, y, z, and $u = x + y - z$ are positive integers that satisfy the equations in the problem statement. Thus if x, y, z and u are positive integers that satisfy the equations in the statement, and $x \geq y$, then $x/y > 3+2\sqrt{2}$. Furthermore, this lower bound is best possible.

Problem N3. Let $a_1 = 11^{11}$, $a_2 = 12^{12}$, $a_3 = 13^{13}$, and

$$a_n = |a_{n-1} - a_{n-2}| + |a_{n-2} - a_{n-3}|, \qquad n \geq 4.$$

Determine $a_{14^{14}}$.

Solution. For $n \geq 2$, define $s_n = |a_n - a_{n-1}|$. Then for $n \geq 5$, $a_n = s_{n-1} + s_{n-2}$ and $a_{n-1} = s_{n-2} + s_{n-3}$, and hence $s_n = |s_{n-1} - s_{n-3}|$. Because $s_n \geq 0$, it follows that if $\max\{s_n,\ s_{n+1},\ s_{n+2}\} \leq T$, then $s_m \leq T$ for all $m \geq n$. In particular, the sequence (s_n) is bounded. We now prove the following claim.

Claim. If $\max\{s_i,\ s_{i+1},\ s_{i+2}\} = T \geq 2$ for some i, then $\max\{s_{i+6},\ s_{i+7},\ s_{i+8}\} \leq T - 1$.

Proof. If this were not the case, then $\max\{s_j,\ s_{j+1},\ s_{j+2}\} = T \geq 2$ for $j = i, i+1, i+2, \ldots, i+6$. We show by contradiction that this cannot happen. If the claim were false, then for $j = i, i+1$, or $i+2$ the sequence $s_j, s_{j+1}, s_{j+2}, \ldots$ would have the form

$$T,\ x,\ y,\ T - y, \ldots,$$

with $0 \leq x,\ y \leq T$, and $\max\{x, y, T - y\} = T$. Hence either $x = T$ or $y = T$ or $y = 0$. We consider each case:

(a) If $x = T$, then the sequence has the form $T, T, y, T - y, y, \ldots$
Because $\max\{y, T - y, y\} = T$ we must have $y = 0$ or $y = T$.

(b) If $y = 0$, then the sequence takes the form $T, x, 0, T, T - x, T - x, x, \ldots$
Hence $\max\{x, T - x\} = T$, so $x = 0$ or $x = T$.

(c) If $y = T$, then the sequence is $T, x, T, 0, x, T - x, \ldots$
We then have $\max\{x, T - x\} = T$, so $x = 0$ or $x = T$.

In each case we find that both x and y must be either 0 or T. In particular, T must divide each of s_i, s_{i+1}, and s_{i+2}, which implies that T divides s_n for all $n \geq 2$. However, because $s_2 = 12^{12} - 11^{11}$ and $s_4 = 11^{11}$ are relatively prime we have a contradiction. This establishes the claim. $\square$

Now let $M = 14^{14}$ and $N = 13^{13}$. From the bound $\max\{s_2, s_3, s_4\} \le N$, we use the claim to deduce inductively that $\max\{s_{6N+2}, s_{6N+3}, s_{6N+4}\} = 1$. In particular $s_n = 0$ or 1 for $n \ge 6N + 2$. Hence $a_n = s_{n-1} + s_{n-2}$ may only take on the values 0, 1, or 2 when $n \ge M > 6N + 4$. In particular, $a_M = 0$, 1, or 2. Now the recursion for a_n implies that

$$a_n \equiv (a_{n-1} - a_{n-2}) + (a_{n-2} - a_{n-3}) \equiv a_{n-1} - a_{n-3} \pmod 2.$$

From the initial values for a_1, a_2, and a_3 it can be easily shown that modulo 2, the sequence (a_n) is periodic with period 7, and that a_7 is odd. Because 14^{14} is a multiple of 7, we conclude that $a_M = 1$.

Problem N4. Let $p \geq 5$ be a prime number. Prove that there exists an integer a with $1 \leq a \leq p-2$ such that neither $a^{p-1}-1$ nor $(a+1)^{p-1}-1$ is divisible by p^2.

Solution. Let $S = \{1, 2, \ldots, p-1\}$ and let $A = \{a \in S : a^{p-1} \not\equiv 1 \pmod{p^2}\}$. We prove that $|A| \geq (p-1)/2$, where $|A|$ denotes the number of elements in A. Indeed, if $1 \leq a \leq p-1$, then the Binomial Theorem gives

$$(p-a)^{p-1} - a^{p-1} \equiv -(p-1)a^{p-2}p \not\equiv 0 \pmod{p^2}.$$

Thus at least one of a and $p-a$ is in A. In particular, $p-1 \in A$ because $1 \notin A$. Now let $p = 2k+1$, $k \geq 2$, and consider the $k-1$ pairs of numbers $\{(2,3), (4,5), \ldots, (2k-2, 2k-1)\}$. If there exists an i, $1 \leq i \leq k-1$, such that $2i \in A$ and $2i+1 \in A$, then the conditions of the problem are satisfied. If not, then at least one entry of each pair $(2i, 2i+1)$, $1 \leq i \leq k-1$, is in A. Because $1 \notin A$ and $|A| \geq (p-1)/2 = k$, exactly one element of each such pair is in A. Consider now the pair $(2k-2, 2k-1)$.

If $2k-1 = p-2 \in A$ then we are done because $p-1 \in A$. (Note that this is the case if $p = 5$.) If $2k-1 = p-2 \notin A$, then $p-3 = 2k-2 \in A$, so the conditions of the problem will be satisfied if we show that $2k-3 \in A$. Suppose also that $2k-3 \notin A$. Because $p-2 \notin A$ we have

$$1 \equiv (p-2)^{p-1} \equiv 2^{p-1} - (p-1)2^{p-2}p \equiv 2^{p-1} + p2^{p-2} \pmod{p^2}. \tag{1}$$

Squaring the first and last expressions in (1), we obtain

$$4^{p-1} + p2^{2p-2} \equiv 1 \pmod{p^2}. \tag{2}$$

Also, because $2k-3 = p-4 \notin A$,

$$1 \equiv (p-4)^{p-1} \equiv 4^{p-1} - (p-1)4^{p-2}p \equiv 4^{p-1} + p4^{p-2} \pmod{p^2}. \tag{3}$$

Subtracting the first and last expressions of (3) from (2), we obtain $3p4^{p-2} \equiv 0 \pmod{p^2}$, a contradiction. Thus, if $p \geq 7$ and $p-2 \notin A$, then $p-4 \in A$. In particular, if $p-3 \in A$, then $p-4$ and $p-3$ satisfy the conditions of the problem.

Problem N5. Let $a > b > c > d$ be positive integers and suppose

$$ac + bd = (b + d + a - c)(b + d - a + c).$$

Prove that $ab + cd$ is not prime.

Solution 1. Suppose to the contrary that $ab + cd$ is prime. Note that

$$ab + cd = (a + d)c + (b - c)a = m \cdot \gcd(a + d, b - c)$$

for some positive integer m. By assumption, either $m = 1$ or $\gcd(a + d, b - c) = 1$. We consider these alternatives in turn.

Case (i): $m = 1$. Then

$$\begin{aligned}\gcd(a + d, b - c) &= ab + cd > ab + cd - (a - b + c + d)\\ &= (a + d)(c - 1) + (b - c)(a + 1)\\ &\geq \gcd(a + d, b - c),\end{aligned}$$

which is false.

Case (ii): $\gcd(a + d, b - c) = 1$. Substituting $ac + bd = (a + d)b - (b - c)a$ for the left-hand side of $ac + bd = (b + d + a - c)(b + d - a + c)$, we obtain

$$(a + d)(a - c - d) = (b - c)(b + c + d).$$

In view of this, there exists a positive integer k such that

$$\begin{aligned}a - c - d &= k(b - c),\\ b + c + d &= k(a + d).\end{aligned}$$

Adding these equations, we obtain $a + b = k(a + b - c + d)$ and thus $k(c - d) = (k - 1)(a + b)$. Recall that $a > b > c > d$. If $k = 1$ then $c = d$, a contradiction. If $k \geq 2$ then

$$2 \geq \frac{k}{k - 1} = \frac{a + b}{c - d} > 2,$$

a contradiction.

Since a contradiction is reached in both (i) and (ii), $ab + cd$ is not prime.

Solution 2. The equality $ac + bd = (b + d + a - c)(b + d - a + c)$ is equivalent to

$$a^2 - ac + c^2 = b^2 + bd + d^2. \tag{1}$$

Let $ABCD$ be the quadrilateral with $AB = a$, $BC = d$, $CD = b$, $AD = c$, $\angle BAD = 60°$, and $\angle BCD = 120°$. Such a quadrilateral exists in view of (1) and the Law of Cosines; the common value in (1) is BD^2. Let $\angle ABC = \alpha$, so that $\angle CDA = 180° - \alpha$. Applying the Law of Cosines to triangles ABC and ACD gives

$$a^2 + d^2 - 2ad \cos\alpha = AC^2 = b^2 + c^2 + 2bc \cos\alpha.$$

Hence $2\cos\alpha = (a^2 + d^2 - b^2 - c^2)/(ad + bc)$, and

$$AC^2 = a^2 + d^2 - ad\,\frac{a^2 + d^2 - b^2 - c^2}{ad + bc} = \frac{(ab + cd)(ac + bd)}{ad + bc}.$$

Because $ABCD$ is cyclic, Ptolemy's Theorem gives

$$(AC \cdot BD)^2 = (ab + cd)^2$$

It follows that

$$(ac + bd)(a^2 - ac + c^2) = (ab + cd)(ad + bc). \tag{2}$$

(*Note.* Also straightforward algebra can be used obtain (2) from (1).) Next observe that

$$ab + cd > ac + bd > ad + bc. \tag{3}$$

The first inequality follows from $(a-d)(b-c)>0$, and the second from $(a-b)(c-d)>0$.

Now assume that $ab + cd$ is prime. It then follows from (3) that $ab + cd$ and $ac + bd$ are relatively prime. Hence, from (2), it must be true that $ac + bd$ divides $ad + bc$. However, this is impossible by (3). Thus $ab + cd$ must not be prime.

Note. Examples of 4-tuples (a, b, c, d) that satisfy the given conditions are $(21, 18, 14, 1)$ and $(65, 50, 34, 11)$.

Problem N6. Is it possible to find 100 positive integers not exceeding 25,000, such that all pairwise sums of them are different?

Solution. Yes. The desired result is an immediate consequence of the following fact.

Lemma. For any odd prime number p there exist p positive integers not exceeding $2p^2$ for which the pairwise sums of the integers are all different.

Proof. Consider the p numbers $f_n = 2pn + (n^2)$, $n = 0, 1, 2, \ldots, p-1$, where (a^2) denotes the remainder when a^2 is divided by p. Because $0 \leq (a^2) \leq p-1$,

$$\left\lfloor \frac{f_m + f_n}{2p} \right\rfloor = m + n. \tag{$*$}$$

Assume that $f_m + f_n = f_k + f_l$. From $(*)$ it follows that $m + n = k + l$ and hence that $(m^2) + (n^2) = (k^2) + (l^2)$, that is, $m^2 + n^2 \equiv k^2 + l^2 \pmod{p}$. The conditions

$$n + m \equiv k + l \pmod{p}, \qquad n^2 + m^2 \equiv k^2 + l^2 \pmod{p},$$

$0 \leq n, m, k, l \leq p-1$, imply that the pairs $\{m, n\}$ and $\{k, l\}$ are the same. Thus, if $\{m, n\} \neq \{k, l\}$, then $f_n + f_m \neq f_k + f_l$. This completes the proof of the lemma. □

Applying the lemma with $p = 101$, we obtain a set of 101 numbers not exceeding $2 \cdot 101^2 < 25000$, all of whose pairwise sums are different.