Problems

Each problem is worth seven points.

Problem 1

Let $ABC$ be an acute-angled triangle with circumcentre $O$. Let $P$ on $BC$ be the foot of the altitude from $A$.

Suppose that $\angle BCA \geq \angle ABC + 30^\circ$.

Prove that $\angle CAB + \angle COP < 90^\circ$.

Problem 2

Prove that

$$\frac{a}{\sqrt{a^2 + 8 \ b \ c}} + \frac{b}{\sqrt{b^2 + 8 \ c \ a}} + \frac{c}{\sqrt{c^2 + 8 \ a \ b}} \geq 1$$

for all positive real numbers $a, b$ and $c$.

Problem 3

Twenty-one girls and twenty-one boys took part in a mathematical contest.

- Each contestant solved at most six problems.
- For each girl and each boy, at least one problem was solved by both of them.

Prove that there was a problem that was solved by at least three girls and at least three boys.

Problem 4

Let $n$ be an odd integer greater than 1, and let $k_1, k_2, \ldots, k_n$ be given integers. For each of the $n!$ permutations $a = (a_1, a_2, \ldots, a_n)$ of $1, 2, \ldots, n$, let

$$S(a) = \sum_{i=1}^{n} k_i \ a_i.$$  

Prove that there are two permutations $b$ and $c, b \neq c$, such that $n!$ is a divisor of $S(b) - S(c)$.
Problem 5

In a triangle $ABC$, let $AP$ bisect $\angle BAC$, with $P$ on $BC$, and let $BQ$ bisect $\angle ABC$, with $Q$ on $CA$.

It is known that $\angle BAC = 60^\circ$ and that $AB + BP = AQ + QB$.

What are the possible angles of triangle $ABC$?

Problem 6

Let $a, b, c, d$ be integers with $a > b > c > d > 0$. Suppose that

$$ac + bd = (b + d + a - c)(b + d - a + c).$$

Prove that $ab + cd$ is not prime.
Problems with Solutions

Problem 1

Let $ABC$ be an acute-angled triangle with circumcentre $O$. Let $P$ on $BC$ be the foot of the altitude from $A$.

Suppose that $\angle BCA \geq \angle ABC + 30^\circ$.

Prove that $\angle CAB + \angle COP < 90^\circ$.

Solution

\begin{itemize}
  \item Solution 1
  \begin{itemize}
    \item Let $\alpha = \angle CAB$, $\beta = \angle ABC$, $\gamma = \angle BCA$, and $\delta = \angle COP$. Let $K$ and $Q$ be the reflections of $A$ and $P$, respectively, across the perpendicular bisector of $BC$. Let $R$ denote the circumradius of $\triangle ABC$. Then $OA = OB = OC = OK = R$.
    \end{itemize}
  \item Furthermore, we have $QP = KA$ because $KQPA$ is a rectangle. Now note that $\angle AOK = \angle AOB - \angle KOB = \angle AOB - \angle AOC = 2\gamma - 2\beta \geq 60^\circ$.
  \end{itemize}

It follows from this and from $OA = OK = R$ that $KA \geq R$ and $QP \geq R$. Therefore, using the Triangle Inequality, we have $OP + R = QO + QC > QC = QP + PC \geq R + PC$. It follows that $OP > PC$, and hence in $\triangle COP$, $\angle PCO > \delta$.

Now since $\alpha = \frac{1}{2} \angle BOC = \frac{1}{2} (180^\circ - 2 \angle PCO) = 90^\circ - \angle PCO$, it indeed follows that $\alpha + \delta < 90^\circ$.

\begin{itemize}
  \item Solution 2
  \begin{itemize}
    \item As in the previous solution, it is enough to show that $OP > PC$. To this end, recall that by the (Extended) Law of Sines, $AB = 2R \sin \gamma$ and $AC = 2R \sin \beta$. Therefore, we have
      \[ BP - PC = AB \cos \beta - AC \cos \gamma = 2R (\sin \gamma \cos \beta - \sin \beta \cos \gamma) = 2R \sin (\gamma - \beta). \]
    \end{itemize}
  \item It follows from this and from
    \[ 30^\circ \leq \gamma - \beta < \gamma < 90^\circ \]
    that $BP - PC \geq R$. Therefore, we obtain that $R + OP = BO + OP > BP \geq R + PC$, from which $OP > OC$, as desired.
  \end{itemize}
Solution 3

We first show that \( R^2 > CP \cdot CB \). To this end, since \( CB = 2R \sin \alpha \) and \( CP = AC \cos \gamma = 2R \sin \beta \cos \gamma \), it suffices to show that \( \frac{1}{4} > \sin \alpha \sin \beta \cos \gamma \). We note that \( 1 > \sin \alpha = \sin(\gamma + \beta) = \sin \gamma \cos \beta + \sin \beta \cos \gamma \) and \( \frac{1}{4} \leq \sin(\gamma - \beta) = \sin \gamma \cos \beta - \sin \beta \cos \gamma \). Since \( 30^\circ \leq \gamma - \beta < 90^\circ \). It follows that \( \frac{1}{4} > \sin \beta \cos \gamma \) and that \( \frac{1}{4} > \sin \alpha \sin \beta \cos \gamma \).

Now we choose a point \( J \) on \( BC \) so that \( CJ \cdot CP = R^2 \). It follows from this and from \( R^2 > CP \cdot CB \) that \( CJ > CB \), so that \( \angle OBC > \angle OJC \). Since \( OC \cdot CJ = PC \cdot CO \) and \( \angle JCO = \angle OCP \), we have \( \triangle JCO \cong \triangle OCP \) and \( \angle OJC = \angle POC = \delta \). It follows that \( \delta < \angle OBC = 90^\circ - \alpha \) or \( \alpha + \delta < 90^\circ \).

Solution 4

On the one hand, as in the third solution, we have \( R^2 > CP \cdot CB \). On the other hand, the power of \( P \) with respect to the circumcircle of \( \triangle ABC \) is \( BP \cdot PC = R^2 - OP^2 \). From these two equations we find that

\[
OP^2 = R^2 - BP \cdot PC > PC \cdot CB - BP \cdot PC = PC^2,
\]

from which \( OP > PC \). Therefore, as in the first solution, we conclude that \( \alpha + \delta < 90^\circ \).

Problem 2

Prove that

\[
\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq 1
\]

for all positive real numbers \( a, b \) and \( c \).

Solution

First we shall prove that

\[
\frac{a}{\sqrt{a^2 + 8bc}} \geq \frac{\frac{a}{3^\frac{1}{2}}}{\frac{a^\frac{1}{2}}{3^\frac{1}{2}} + \frac{b^\frac{1}{2}}{3^\frac{1}{2}} + \frac{c^\frac{1}{2}}{3^\frac{1}{2}}},
\]

or equivalently, that

\[
\left(\frac{a^\frac{1}{2}}{3^\frac{1}{2}} + \frac{b^\frac{1}{2}}{3^\frac{1}{2}} + \frac{c^\frac{1}{2}}{3^\frac{1}{2}}\right)^2 \geq \frac{a}{3^\frac{1}{2}}(a^2 + 8bc).
\]

The AM-GM inequality yields

\[
\left(\frac{a^\frac{1}{2}}{3^\frac{1}{2}} + \frac{b^\frac{1}{2}}{3^\frac{1}{2}} + \frac{c^\frac{1}{2}}{3^\frac{1}{2}}\right)^2 - \left(\frac{a^\frac{1}{2}}{3^\frac{1}{2}}\right)^2 = \left(\frac{b^\frac{1}{2}}{3^\frac{1}{2}} + \frac{c^\frac{1}{2}}{3^\frac{1}{2}}\right)\left(\frac{a^\frac{1}{2}}{3^\frac{1}{2}} + \frac{a^\frac{1}{2}}{3^\frac{1}{2}} + \frac{b^\frac{1}{2}}{3^\frac{1}{2}} + \frac{c^\frac{1}{2}}{3^\frac{1}{2}}\right)
\geq 2\frac{b^\frac{1}{2}}{3^\frac{1}{2}} \cdot \frac{c^\frac{1}{2}}{3^\frac{1}{2}} \cdot 4\frac{a^\frac{1}{2}}{3^\frac{1}{2}} \cdot \frac{b^\frac{1}{2}}{3^\frac{1}{2}} \cdot \frac{c^\frac{1}{2}}{3^\frac{1}{2}}
= 8a \frac{2}{3} b c.
\]

Thus
\[
\left( a^{\frac{4}{5}} + b^{\frac{4}{5}} + c^{\frac{4}{5}} \right)^2 \geq \left( a^{\frac{4}{5}} \right)^2 + 8 a^{\frac{2}{5}} b c = a^{\frac{2}{5}} (a^2 + 8 b c),
\]

so

\[
\frac{a}{\sqrt{a^2 + 8 b c}} \geq \frac{a^{\frac{4}{5}}}{a^{\frac{2}{5}} + b^{\frac{2}{5}} + c^{\frac{2}{5}}}
\]

Similarly, we have

\[
\frac{b}{\sqrt{b^2 + 8 c a}} \geq \frac{b^{\frac{4}{5}}}{a^{\frac{2}{5}} + b^{\frac{2}{5}} + c^{\frac{2}{5}}}
\]

and

\[
\frac{c}{\sqrt{c^2 + 8 a b}} \geq \frac{c^{\frac{4}{5}}}{a^{\frac{2}{5}} + b^{\frac{2}{5}} + c^{\frac{2}{5}}}
\]

Adding these three inequalities yields

\[
\frac{a}{\sqrt{a^2 + 8 b c}} + \frac{b}{\sqrt{b^2 + 8 c a}} + \frac{c}{\sqrt{c^2 + 8 a b}} \geq 1.
\]

**Comment.** It can be shown that for any \( a, b, c > 0 \) and \( \lambda \geq 8 \), the following inequality holds:

\[
\frac{a}{\sqrt{a^2 + \lambda b c}} + \frac{b}{\sqrt{b^2 + \lambda c a}} + \frac{c}{\sqrt{c^2 + \lambda a b}} \geq \frac{3}{\sqrt{1 + \lambda}}.
\]

**Problem 3**

Twenty-one girls and twenty-one boys took part in a mathematical contest.

- Each contestant solved at most six problems.
- For each girl and each boy, at least one problem was solved by both of them.

Prove that there was a problem that was solved by at least three girls and at least three boys.

**Solution**

- **Solution 1**

  We introduce the following symbols: \( G \) is the set of girls at the competition and \( B \) is the set of boys, \( P \) is the set of problems, \( P(g) \) is the set of problems solved by \( g \in G \), and \( P(b) \) is the set of problems solved by \( b \in B \). Finally, \( G(p) \) is the set of girls that solve \( p \in P \) and \( B(p) \) is the set of boys that solve \( p \). In terms of this notation, we have that for all \( g \in G \) and \( b \in B \),

  \[
  (a) \quad |P(g)| \leq 6, \quad |P(b)| \leq 6, \quad (b) \quad P(g) \cap P(b) \neq \emptyset.
  \]

  We wish to prove that some \( p \in P \) satisfies \( |G(p)| \geq 3 \) and \( |B(p)| \geq 3 \). To do this, we shall assume the contrary and reach a contradiction by counting (two ways) all ordered triples \((p, q, r)\) such that \( p \in P(g) \cap P(b) \). With \( T = \{(p, g, b) : p \in P(g) \cap P(b)\} \), condition (b) yields
\[ |T| = \sum \sum_{b \in B} |P(g) \cap P(b)| \geq |G| \cdot |B| = 21^2. \]  \hspace{1cm} (1)

Assume that no \( p \in P \) satisfies \( |G(p)| \geq 3 \) and \( |B(p)| \geq 3 \). We begin by noting that

\[ \sum_{p \in P} |G(p)| = \sum_{g \in G} |P(g)| \leq 6 |G| \quad \text{and} \quad \sum_{p \in P} |B(p)| \leq |B|. \]  \hspace{1cm} (2)

(Note. The equality in (2) is obtained by a standard double-counting technique. Let \( \chi(g, p) = 1 \) if \( g \) solves \( p \) and \( \chi(g, p) = 0 \) otherwise, and interchange the orders of summation in \( \sum_{p \in P} \sum_{g \in G} \chi(g, p). \) Let

\[ P_+ = \{ p \in P : |G(p)| \geq 3 \}, \]
\[ P_- = \{ p \in P : |G(p)| \leq 2 \}. \]

**Claim.** \( \sum_{p \in P_-} |G(p)| \geq |G|; \) thus \( \sum_{p \in P_+} |G(p)| \leq 5 |G| \). Also \( \sum_{p \in P_+} |B(p)| \geq |B| \); thus \( \sum_{p \in P_-} |B(p)| \leq 5 |B| \).

**Proof.** Let \( g \in G \) be arbitrary. By the Pigeonhole Principle, conditions (a) and (b) imply that \( g \) solves some problem \( p \) that is solved by at least \( [21/6] = 4 \) boys. By assumption, \( |B(p)| \geq 4 \) implies that \( p \in P_- \), so every girl solves at least one problem in \( P_- \). Thus

\[ \sum_{p \in P_-} |G(p)| \geq |G|. \]  \hspace{1cm} (3)

In view of (2) and (3) we have

\[ \sum_{p \in P_+} |G(p)| = \sum_{p \in P} |G(p)| - \sum_{p \in P_-} |G(p)| \leq 5 |G|. \]

Also, each boy solves a problem that is solved by at least four girls, so each boy solves a problem \( p \in P_+ \). Thus \( \sum_{p \in P_+} |B(p)| \geq |B| \), and the calculation proceeds as before using (2). \( \square \)

Using the claim just established, we find

\[ |T| = \sum_{p \in P} |G(p)| \cdot |B(p)| \]
\[ = \sum_{p \in P_+} |G(p)| \cdot |B(p)| + \sum_{p \in P_-} |G(p)| \cdot |B(p)| \]
\[ \leq 2 \sum_{p \in P_+} |G(p)| + 2 \sum_{p \in P_-} |B(p)| \]
\[ \leq 10 |G| + 10 |B| = 20 \cdot 21. \]

This contradicts (1), so the proof is complete.
Solution 2

Let us use some of the notation given in the first solution. Suppose that for every $p \in P$ either $|G(p)| \leq 2$ or $|B(p)| \leq 2$. For each $p \in P$, color $p$ red if $|G(p)| \leq 2$ and otherwise color it black. In this way, if $p$ is red then $|G(p)| \leq 2$ and if $p$ is black then $|B(p)| \leq 2$. Consider a chessboard with 21 rows, each representing one of the girls, and 21 columns, each representing one of the boys. For each $g \in G$ and $b \in B$, color the square corresponding to $(g, b)$ as follows: pick $p \in P(g) \cap P(b)$ and assign $p$’s color to that square. (By condition (b), there is always an available choice.) By the Pigeonhole Principle, one of the two colors is assigned to at least $[441/2] = 221$ squares, and thus some row has at least $[221/21] = 11$ black squares or some column has at least 11 red squares.

Suppose the row corresponding to $g \in G$ has at least 11 black squares. Then for each of 11 squares, the black problem that was chosen in assigning the color was solved by at most 2 boys. Thus we account for at least

$$\sum_{i=1}^{g} k_{i} a_{i}.$$ 

In view of condition (a), $g$ solves only these problems. But then at most 12 boys solve a problem also solved by $g$, in violation of condition (b).

Hence some $p \in P$ satisfies $|G(p)| \geq 3$ and $|B(p)| \geq 3$.

Problem 4

Let $n$ be an odd integer greater than 1, and let $k_1, k_2, \ldots, k_n$ be given integers. For each of the $n!$ permutations $a = (a_1, a_2, \ldots, a_n)$ of $1, 2, \ldots, n$, let

$$S(a) = \sum_{i=1}^{n} k_{i} a_{i}.$$ 

Prove that there are two permutations $b$ and $c$, $b \neq c$, such that $n!$ is a divisor of $S(b) - S(c)$.

Solution

Let $\sum S(a)$ be the sum of $S(a)$ over all $n!$ permutations $a = (a_1, a_2, \ldots, a_n)$. We compute $\sum S(a)$ mod $n!$ two ways, one of which depends on the desired conclusion being false, and reach a contradiction when $n$ is odd.

First way. In $\sum S(a), k_1$ is multiplied by each $i \in \{1, \ldots, n\}$ a total of $(n-1)!$ times, once for each permutation of $\{1, \ldots, n\}$ in which $a_1 = i$. Thus the coefficient of $k_1$ in $\sum S(a)$ is

$$(n-1)! (1 + 2 + \cdots + n) = (n+1)!/2.$$ 

The same is true for all $k_i$, so

$$\sum S(a) = \frac{(n+1)!}{2} \sum_{i=1}^{n} k_{i}. \quad (1)$$

Second way. If $n!$ is not a divisor of $S(b) - S(c)$ for any $b \neq c$, then each $S(a)$ must have a different remainder mod $n!$. Since there are $n!$ permutations, these remainders must be precisely the numbers $0, 1, 2, \ldots, n! - 1$. Thus

$$\sum S(a) \equiv \frac{(n! - 1)n!}{2} \mod n!. \quad (2)$$

Combining (1) and (2), we get

$$\frac{(n+1)!}{2} \sum_{i=1}^{n} k_{i} \equiv \frac{(n! - 1)n!}{2} \mod n!. \quad (3)$$
Now, for $n$ odd, the left side of (3) is congruent to 0 modulo $n!$, while for $n > 1$ the right side is not congruent to 0 ($n! - 1$ is odd). For $n > 1$ and odd, we have a contradiction.

**Problem 5**

In a triangle $ABC$, let $AP$ bisect $\angle BAC$, with $P$ on $BC$, and let $BQ$ bisect $\angle ABC$, with $Q$ on $CA$.

It is known that $\angle BAC = 60^\circ$ and that $AB + BP = AQ + QB$.

What are the possible angles of triangle $ABC$?

**Solution**

Denote the angles of $ABC$ by $\alpha = 60^\circ$, $\beta$, and $\gamma$. Extend $AB$ to $P'$ so that $BP' = BP$, and construct $P''$ on $AQ$ so that $AP'' = AP'$. Then $BP'P$ is an isosceles triangle with base angle $\beta/2$. Since $AQ + QP'' = AB + BP' = AB + BP = AQ + QB$, it follows that $QP'' = QB$. Since $AP'P''$ is equilateral and $AP$ bisects the angle at $A$, we have $PP'' = PP'$.

Claim. Points $B$, $P$, $P''$ are collinear, so $P''$ coincides with $C$.

*Proof.* Suppose to the contrary that $BPP''$ is a nondegenerate triangle. We have that $\angle PBQ = \angle PP'B = \angle PP''Q = \beta/2$. Thus the diagram appears as below, or else with $P$ is on the other side of $BP''$. In either case, the assumption that $BPP''$ is nondegenerate leads to $BP = PP'' = PP'$, thus to the conclusion that $BPP''$ is equilateral, and finally to the absurdity $\beta/2 = 60^\circ$ so $\alpha + \beta = 60^\circ + 120^\circ = 180^\circ$. 

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Thus points $B$, $P$, $P''$ are collinear, and $P'' = C$ as claimed. □

Since triangle $BCQ$ is isosceles, we have $120^\circ - \beta = \gamma = \beta / 2$, so $\beta = 80^\circ$ and $\gamma = 40^\circ$. Thus $ABC$ is a 60-80-40 degree triangle.

**Problem 6**

Let $a$, $b$, $c$, $d$ be integers with $a > b > c > d > 0$. Suppose that

$$a c + b d = (b + d + a - c) (b + d - a + c).$$

Prove that $a b + c d$ is not prime.

**Solution**

- **Solution 1**

  Suppose to the contrary that $a b + c d$ is prime. Note that

  $$a b + c d = (a + d) c + (b - c) a = m \cdot \gcd(a + d, b - c)$$

  for some positive integer $m$. By assumption, either $m = 1$ or $\gcd(a + d, b - c) = 1$. We consider these alternatives in turn.

  **Case (i):** $m = 1$. Then

  $$\gcd(a + d, b - c) = a b + c d > a b + c d - (a - b + c + d)$$

  $$= (a + d) (c - 1) + (b - c) (a + 1)$$

  $$\geq \gcd(a + d, b - c),$$

  which is false.

  **Case (ii):** $\gcd(a + d, b - c) = 1$. Substituting $a c + b d = (a + d) b - (b - c) a$ for the left-hand side of $a c + b d = (b + d + a - c) (b + d - a + c)$, we obtain

  $$(a + d) (a - c - d) = (b - c) (b + c + d).$$

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In view of this, there exists a positive integer \( k \) such that

\[
\begin{align*}
    a - c - d &= k(b - c), \\
    b + c + d &= k(a + d).
\end{align*}
\]

Adding these equations, we obtain \( a + b = k(a + b - c + d) \) and thus \( k(c - d) = (k - 1)(a + b) \). Recall that \( a > b > c > d \). If \( k = 1 \) then \( c = d \), a contradiction. If \( k \geq 2 \) then

\[
2 \geq \frac{k}{k - 1} = \frac{a + b}{c - d} > 2,
\]

a contradiction.

Since a contradiction is reached in both (i) and (ii), \( a \ b \ + \ c \ d \) is not prime.

\[\blacktriangleleft\]

**Solution 2**

The equality \( a \ c + b \ d = (b + d + a - c)(b + d - a + c) \) is equivalent to

\[
a^2 - a \ c + c^2 = b^2 + b \ d + d^2.
\]  

(1)

Let \( ABCD \) be the quadrilateral with \( AB = a \), \( BC = d \), \( CD = b \), \( AD = c \), \( \angleBAD = 60^\circ \), and \( \angleBCD = 120^\circ \). Such a quadrilateral exists in view of (1) and the Law of Cosines; the common value in (1) is \( BD^2 \). Let \( \angleABC = \alpha \), so that \( \angleCDA = 180^\circ - \alpha \). Applying the Law of Cosines to triangles \( ABC \) and \( ACD \) gives

\[
a^2 + d^2 - 2 \ a \ d \ \cos \alpha = A \ C^2 = b^2 + c^2 + 2 \ b \ c \ \cos \alpha.
\]

Hence \( \cos \alpha = (a^2 + d^2 - b^2 - c^2)/(a \ d + b \ c) \), and

\[\begin{align*}
A \ C^2 &= a^2 + d^2 - a \ d \ 
\frac{a^2 + d^2 - b^2 - c^2}{a \ d + b \ c} = \frac{(a \ b + c \ d)(a \ c + b \ d)}{a \ d + b \ c}.
\end{align*}\]

Because \( ABCD \) is cyclic, Ptolemy's Theorem gives

\[
(A \ C \cdot B \ D)^2 = (a \ b + c \ d)^2
\]

It follows that

\[
(a \ c + b \ d)(a^2 - a \ c + c^2) = (a \ b + c \ d)(a \ d + b \ c).
\]

(2)

(Note. Straightforward algebra can also be used obtain (2) from (1).) Next observe that

\[
a \ b + c \ d > a \ c + b \ d > a \ d + b \ c.
\]

(3)

The first inequality follows from \((a - d)(b - c) > 0\), and the second from \((a - b)(c - d) > 0\).

Now assume that \( a \ b + c \ d \) is prime. It then follows from (3) that \( a \ b + c \ d \) and \( a \ c + b \ d \) are relatively prime. Hence, from (2), it must be true that \( a \ c + b \ d \) divides \( a \ d + b \ c \). However, this is impossible by (3). Thus \( a \ b + c \ d \) must not be prime.

*Note.* Examples of 4-tuples \( (a, b, c, d) \) that satisfy the given conditions are \((21, 18, 14, 1)\) and \((65, 50, 34, 11)\).