Duration : 4 hr 30 min
7 points each problem

Problem 1

Two circles Γ_1 and Γ_2 intersect at M and N.

Let ℓ be the common tangent to Γ_1 and Γ_2 so that M is closer to ℓ than N is. Let ℓ touch Γ_1 at A and Γ_2 at B. Let the line through M parallel to ℓ meet the circle Γ_1 again at C and the circle Γ_2 again at D.

Lines CA and DB meet at E; lines AN and CD meet at P; lines BN and CD meet at Q.

Show that $EP = EQ$.

Problem 2

Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

Problem 3

Let $n \geq 2$ be a positive integer. Initially, there are n fleas on a horizontal line, not all at the same point.

For a positive real number λ, define a *move* as follows:

> choose any two fleas, at points A and B, with A to the left of B;
>
> let the flea at A jump to the point C on the line to the right of B with $BC/AB = \lambda$.

Determine all values of λ such that, for any point M on the line and any initial positions of the n fleas, there is a finite sequence of moves that will take all the fleas to positions to the right of M.

Duration : 4 hr 30 min
7 points each problem

Problem 4

A magician has one hundred cards numbered 1 to 100. He puts them into three boxes, a red one, a white one and a blue one, so that each box contains at least one card.

A member of the audience selects two of the three boxes, chooses one card from each and announces the sum of the numbers on the chosen cards. Given this sum, the magician identifies the box from which no card has been chosen.

How many ways are there to put all the cards into the boxes so that this trick always works? (Two ways are considered different if at least one card is put into a different box.)

Problem 5

Determine whether or not there exists a positive integer n such that

n is divisible by exactly 2000 different prime numbers, and

$2^n + 1$ is divisible by n.

Problem 6

Let AH_1, BH_2, CH_3 be the altitudes of an acute-angled triangle ABC. The incircle of the triangle ABC touches the sides BC, CA, AB at T_1, T_2, T_3, respectively. Let the lines ℓ_1, ℓ_2, ℓ_3 be the reflections of the lines H_2H_3, H_3H_1, H_1H_2 in the lines T_2T_3, T_3T_1, T_1T_2, respectively.

Prove that ℓ_1, ℓ_2, ℓ_3 determine a triangle whose vertices lie on the incircle of the triangle ABC.