Concerning "Solving Mathematical Problems: A Personal Perspective" by Terence Tao

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Introduction

Terence Tao, 2006 Fields medal winner, wrote a delightful book [6] on problem solving in (elementary) mathematics. It includes an excellent selection of 26 problems with fully documented solutions, That is, Terence not only provides an answer and a proof, but explains in detail the reasoning that led him to the answer. The book also includes a set of exercises (without solutions) to practice the illustrated techniques.

Problems for which full solutions are presented

The following problems —mostly taken from well-known sources— are treated in depth.

Problem 1.1 (p. 1) A triangle has its lengths in an arithmetic progression, with difference d. The area of the triangle is t. Find the lengths and angles of the triangle.

Problem 2.1 (p. 11) Show that among any 18 consecutive 3-digit numbers there is at least one which is divisible by the sum of its digits. [7, p. 7]

Problem 2.2 (p. 14) Is there a power of 2 such that its digits could be re-arranged and made into another power of 2? (No zeroes are allowed in the leading digit: e.g. 0032 is not allowed.) [7, p. 37]

Problem 2.3 (p. 19) Find all integers n such that the equation $1/a + 1/b = n/(a+b)$ is satisfied for some non-zero integer values of a and b (with $a + b \neq 0$). [2, p. 15]

Problem 2.4 (p. 20) Find all solutions of $2^n + 7 = x^2$ where n and x are integers. [7, p. 7]

Problem 2.5 (p. 23) Prove that for any nonnegative integer n, the number $1^n + 2^n + 3^n + 4^n$ is divisible by 5 if and only if n is not divisible by 4. [4, p. 74]

Problem 2.6 (p. 24) (**) Let k, n be natural numbers with k odd. Prove that the sum $1^k + 2^k + \cdots + n^k$ is divisible by $1 + 2 + \cdots + n$. [5, p. 14]

Problem 2.7 (p. 27) Let p be a prime number greater than 3. Show that the numerator of the (reduced) fraction $1/1 + 1/2 + 1/3 + \cdots + 1/(p?1)$ is divisible by p^2. For example, when p is 5, the fraction is $1/1 + 1/2 + 1/3 + 1/4 = 25/12$, and the numerator is obviously divisible by 5^2. [5, p. 17]

Problem 3.1 (p. 36) (*) Suppose f is a function mapping the positive integers to the positive integers, such that f satisfies $f(n+1) > f(f(n))$ for all positive integers n. Show that $f(n) = n$ for all positive integers n. [3, p. 19]

Problem 3.2 (p. 38) Suppose f is a function on the positive integers which takes integer values with the following properties:

(a) $f(2) = 2$

(b) $f(mn) = f(m)f(n)$ for all positive integers m and n

(c) $f(m) > f(n)$ if $m > n$.

Find $f(1983)$ (with reasons, of course). [2, p. 7]

Problem 3.3 (p. 43) Let a, b, c be real numbers such that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{a+b+c}$$

with all denominators non-zero. Prove that

$$\frac{1}{a^5} + \frac{1}{b^5} + \frac{1}{c5} = \frac{1}{(a+b+c)^5}$$

[2, p. 13]

Problem 3.4 (p. 45) (**) Prove that any polynomial of the form $f(x) = (x - a_0)^2(x - a_1)^2 \cdots (x - a_n)^2 + 1$ where $a_0, a_1, \ldots, a_n$ are all integers, cannot be factorized into two non-trivial polynomials, each with integer coefficients.

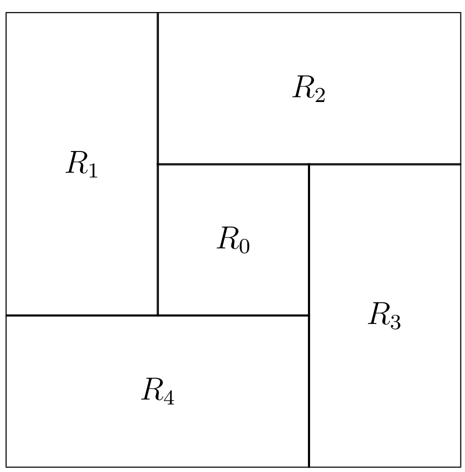
Problem 4.1 (p. 50) ABC is a triangle that is inscribed in a circle. The angle bisectors of A, B, C meet the circle at D, E, F, respectively. Show that AD is perpendicular to EF. [2, p. 12]

Problem 4.2 (p. 52) In triangle BAC the bisector of the angle at B meets AC at D; the angle bisector of C meets AB at E. These bisectors meet at O. Suppose that $|OD| = |OE|$. Prove that either $\angle BAC = 60°$ or that BAC is isosceles (or both). [7, p. 8, Q1]

Problem 4.3 (p. 55) (*) Let $ABFE$ be a rectangle and D be the intersection of the diagonals AF and BE. A straight line through E meets the extended line AB at G and the extended line FB at C so that $|DC| = |DG|$. Show that $|AB|/|FC| = |FC|/|GA| = |GA|/|AE|$. [2, p. 13]

Problem 4.4 (p. 58) Given three parallel liines, construct (with straightedge and compass) an equilateral triangle with each parallel line containing one of the vertices of the triangle.

Problem 4.5 (p. 62) A square is divided into five rectangles as shown below. The four outer rectangles R_1, R_2, R_3, R_4 all have the same area. Prove that the inner rectangle R_0 is a square. [7, p. 10, Q4]



Problem 4.6 (p. 66) Let $ABCD$ be a square, and let k be the circle with centre B passing through A, and let $l$be the semicircle inside the square with diameter AB. Let E be a point on l and let the extension of B meet circle k at F. Prove that $\angle DAF = \angle EAF$. [1, Q1]

Problem 5.1 (p. 69) A regular polygon with n vertices is inscribed in a circle of radius 1. Let L be the set of all possible distinct lengths of all line segments joining the vertices of the polygon. What is the sum of the squares of the elements of L? [2, p. 14]

Problem 5.2 (p. 74) (*) A rectangle is partitioned into several smaller rectangles. Each of the smaller rectangles has at least one side of integer length. Prove that the big rectangle has at least one side of integer length.

Problem 5.3 (p. 77) On a plane we have a finite collection of points, no three of which are collinear. Some points are joined to others by line segments, but each point has at most on line segment attached to it. Now we perform the following procedure: We take two intersecting line segments, say AB and CD, and remove them an replace them with AC and BD. Is it possible to perform this procedure indefinitely? [7, p. 8]

Problem 5.4 (p. 79) In the centre of a square swimming pool is a boy, while his teacher (who cannot swim) is at one corner of the pool. The teacher can run three times faster than the boy can swim, but the boy can run faster than the teacher can. Can the boy escape from the teacher? (Assume both persons are infinitely manoeuvrable.) [7, p. 34, Q2]

Problem 6.1 (p. 83) Suppose on a certain island there are 13 grey, 15 brown, and 17 crimson chameleons. If two chameleons of different colour meet, they both change to the third colour (e.g. a brown and crimson pair would both change to grey). This is the only time they change colour. Is it possible for all chameleons to eventually be the same colour? [7, p. 25, Q5]

Problem 6.2 (p. 86) (*) Alice, Betty, and Carol took the same series of examinations. For each examination there was one mark of x, one mark of y, and one mark of z, where x, y, z are distinct positive integers. After all the examinations, Alice had a total score of 20, Betty a total score of 10, and Carol a total score of 9. If Betty was placed first in Algebra, who was placed second in Geometry?

Problem 6.3 (p. 90) Two people play a game with a bar of chocolate mad of 60 pieces, in a 6×10 rectangle. The first person breaks off a part of the chocolate bar along the grooves dividing the pieces, and discards (eats) the part he broke off. The the second breaks off a part of the remaining part and discards her part. The game continues until one piece is left. The winner is the one who leaves the other with the single piece (i.e. is the last to move). Which person has a perfect winning strategy? [7, p. 16, Q3]

Problem 6.4 (p. 95) Two brothers sold a herd of sheep. Each sheep sold for as many rubles as the number of sheep originally in the herd. The money was then divided in the following manner. First the older borther took 10 rubles, the the younger brother took 10 rubles, then the older brother took another 10 rubbles, and so on. At the end of the division the younger brother, whose turn it was, found that there were fewer than 10 roubles left, so he took what remained. To make the division fair, the older brother gave the younger his penknife, which was worth an integer number of roubles. How much was the penknife worth? [5, p. 9]

Errata and remarks

p. 2, l. 18 "Problem 1.1 question": delete "question"

p. 5, l. 7 "gung-ho": 'Gung-ho is a phrase borrowed from Chinese language, frequently used in Chinese as an adjective meaning *enthusiastic*.' (from Wikipedia)

p. 6, l. –14 "once": change to "one"

p. 7, l. 7 "put clear": delete "put"

p. 7, l. –3 "ompute": change to "compute"

p. 9, l. –9 "which are exactly the same": change "are" to "have" (?)

p. 16, l. –5 "the digit sum of 217 is a mere 14": change "217" to "2^{17}

p. 17, l. 14 "flash": change to "flashy"

p. 25, l. 9 "n": change to "n"

p. 27, Problem 2.7 The given $p > 3$ is used rather late. On p. 30, l. –9, "p is an odd prime" appears, and on p. 33, l. –14, "p is a prime greater than 3" appears.

p. 31, l. 9 "we are now reduced": change to "we have now reduced it to"

p. 33, l. 5 "Thus": why is this inference valid?

p. 34, l. 3 “trick”: I don’t like that terminology

p. 35, l. 4 (of quote) “that”: change to “than”

p. 39, l. 17 “$f(1) = 1$”: follows immediately from (b) and (a): $f(2) = f(2 \cdot 1) = f(2)f(1)$ and $f(2) = 2 \neq 0$.

p. 40, l. 13 “smells heavily on”: change “on” to “of”

p. 42, l. –2 “one has”: change to “these have”

p. 44, l. –14 and –3 “$5ab$”: I don’t understand this; what is the significance/role/purpose of $5ab$?

p. 44 From the solution it follows that Problem 3.3 can be generalized, by replacing the exponent 5 by any odd $n > 0$. The relevance of 5 was already doubtful from the beginning. Note that we have $a + b = 0$ implies $1/a + 1/b = 0$, and hence also $1/a^n + 1/b^n = 0$ for odd $n > 0$.

p. 45, l. 9 “$(x - a_1)^2, \ldots, (x - a_n)^2$”: change “$, \ldots ,$” to “$\cdots$”

p. 46, l. –7 Why is it not possible that $p(x) = 1$ for all x?

p. 47, Exercise 3.7 The problem statement is wrong. One of many counterexamples: $(x-0)(x+2)+1 = (x+1)(x+1)$. The hint suggest that if $f(x) = p(x)q(x)$ then $p(x) \equiv q(x)$, which is not what was asked.

p. 49, Figure It might have helped the unexperienced reader to label $\angle PAO = \angle OPA = \alpha$ and $\angle OPB = \angle PBO = \beta$, and stating that $\alpha+(\alpha+\beta)+\beta = 180°$, hence $\alpha + \beta = 90°$..

p. 50, Figure Label of point D (on BC) is missing.

p. 53, Figure $\angle ADO$ labeled $(\gamma + \beta)/2$: change to $\gamma + \beta/2$
$\angle AEO$ labeled $(\beta + \gamma)/2$: change to $\beta + \gamma/2$

p. 56, l. 2 below (15) “the third ratio”: note that given $|DC| = |DG|$ has not yet been used.

p. 57, Proof “PQT is similar to PTR” may not be obvious. This follows from ‘same chords subtend same angles’ (a picture would help)

p. 58, Problem 4.4 Appeared in [9, p. 31, #18]

p. 58, l. –1 “as best we can”: insert “as” after “best” (?)

p. 61, l. 1 “so long as they are not at 60° angles”: this is not correct; in that case there is only one solution, not two.

p. 63, l. 3 “square to be equal”: change to “rectangle to be a square”

p. 64, l. 4 “a rectangle with a fixed area”: this concerns the outer rectangles

p. 65, l. 3 “seems to suggest that $a + b$ should equal 1”: why?

p. 66, l. –8 “Correspondence Problem”: change “Problem” to “Programme” (?)

p. 66, Figure (nitpicking) The arc AC should not cross side AD

p. 74, l. 10 “of”: change to “if”

p. 74, Problem 5.2 Also see [8]

p. 77, l. –8 “assertion is plausible”: it is not clear what is meant; there is no assertion in the problem statement, only a question; presumably, the assertion that ‘the procedure can*not* be performed indefinitely’ is meant

p. 79, l. 4 “triangle inequality”: the uninitated reader may need some help applying it (translate e.g. AB over vector $\overrightarrow{AC}$ to yield $A'B'$ with $A' = C$; now consider $\triangle DB'C$; a picture would help)

p. 81, l. 6 “not a clever move”: intimidating, not convincing

p. 81, l. 15 “the midpoint M of B and C”: change “C” to “D” (?)

p. 81, l.–6 “BC”: change to “BD”

p. 83, l.–7 “systems”: change to “states” (?)

p. 83, Problem 6.1 Concerning solution: observe that pairwise differences in the number of chameleons modulo 3 are invariant.

p. 86, Problem 6.2 First appeared at IMO 1974 as problem 1.

p. 87, l. -6 “eliminated (c)”: insert “is” before “(c)”

p. 90, l. 13 “ths”: change to “this”

p. 92, l. 22 “that it is”: delete “is”

p. 95, l. 2, 4, 5, 7, 9 "ruble", "rouble": inconsistent spelling (both are correct)

p. 95, Problem 6.4 Not clear whether sheep are sold one-by-one or all at once; From the solution it follows that 'all at once' is intended.

p. 95, l. 16 "in terms of equation": change "equation" to "equations"

p. 95, l. 22 "windfall": 'unexpected or sudden gain or advantage' (according to Webster)

p. 96, l. −1 "is restricted to between 1 and 9": insert "lie" after "to"

References I cannot confirm the reference to Taylor 1989.

References

[1] AMOC (Australian Mathematical Olympiad Committee) *Correspondence Programme, 1986–1987, Set One*.

[2] Australian Mathematics Competition. *Mathematical Olympiads: The 1987 Australian Scene*. Canberra College of Advanced Education, 1987

[3] Samual L. Greitzer. *International Mathematical Olympiads, 1955-1977*. Mathematical Association of America, 1978.

[4] József Kürshák; G. Hajós, G. Neukomm, L. Surányi (Eds.). *Hungarian Problem Book I, based on the Eötvös competitions 1894–1905*. Mathematical Association of America, 1963/1967.

[5] D. O. Shklarsky, N. N. Chentzov, I. M. Yaglom, (I. Sussman, ed.). *The USSR Olympiad Problem Book: Selected Problems and Theorems in Elementary Mathematics*, Freeman, 1962.

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[7] Peter J. Taylor. *International Mathematics Tournament of the Towns 1984–1989: Questions and Solutions*. Australian Mathematics Trust Publishing, 1989. (A 2nd corrected edition is available.)

[8] Stan Wagon. "Fourteen Proofs of a Result about Tiling a Rectangle", *American Mathematical Monthly*, **94**(14):601–617 (Aug-Sep 1987).

[9] I. M. Yaglom. *Geometric Transformations I*. Mathematical Association of America, 1962.